REAL C^k KOEBE PRINCIPLE

WEIXIAO SHEN AND MICHAEL TODD

ABSTRACT. We prove a C^k version of the real Koebe principle for interval (or circle) maps with non-flat critical points.

1. Introduction

The real Koebe principle, providing estimates of the first derivative of iterates of a smooth interval map, plays a very important role in recent research of one-dimensional dynamics. See [**MS**]. Considering its complex counterpart, the (*complex*) Koebe distortion theorem, it is natural to look for a C^k, $k \geq 2$, version of this principle. This is the goal of this paper.

More precisely, let f be a C^k endomorphism of the compact interval $I = [0, 1]$ (or the circle $\mathbb{R}/\mathbb{Z}$). We assume that f has only non-flat critical points; that is, for each critical point c of f, there exists a real number $\alpha > 1$, such that under some C^k coordinate change, we have

$$|f(x) - f(c)| = |x|^\alpha$$

for x close to c. We use NF^k to denote the class of such maps.

Theorem 1. *Let f be in the class NF^n, $n \geq 2$. Let T be an open interval such that $f^s : T \to f^s(T)$ is a diffeomorphism. For each $S, \kappa > 0$ and each $1 \leq k \leq n$ there exist $\delta = \delta(S, \kappa, f) > 0$ and $K_k = K_k(\kappa) > 0$ satisfying the following. If $\sum_{j=0}^{s-1} |f^j(T)| \leq S$ and J is a closed subinterval of T such that*

- *$f^s(T)$ is a κ-scaled neighbourhood of $f^s(J)$;*
- *$|f^j(J)| < \delta$ for $0 \leq j < s$,*

then, letting $\psi_0 : J \to I$ and $\psi_s : f^s(J) \to I$ be affine diffeomorphisms, we have

$$\|\psi_s f^s \psi_0^{-1}\|_{C^k} < K_k.$$

Furthermore, $K_1 \to 1$ as $\kappa \to \infty$ and for each $k > 1$, $K_k \to 0$ as $\kappa \to \infty$.

The well-known real Koebe principle claims the existence of K_1.

1.1. **Proof of Theorem 1.**

To prove this theorem, we shall approximate the map $\psi_s f^s \psi_0^{-1}$ by maps in the Epstein class, and then apply the (complex) Koebe distortion theorem. The main step is to prove the following theorem.

2000 *Mathematics Subject Classification.* 37E05.

Theorem 2. *Let f be a map in the class NF^n, $n = 2, 3 \ldots$. Let T be an open interval such that $f^s : T \to f^s(T)$ is a diffeomorphism. For any $S, \kappa, \varepsilon > 0$ and $1 \leq k \leq n$, there exist $\delta = \delta(S, \kappa, \varepsilon) > 0$ and $\beta = \beta(\kappa) > 0$ satisfying the following. Suppose that $\sum_{j=0}^{s-1} |f^j(T)| \leq S$ and J is a closed subinterval of T such that*

- *$f^s(T)$ is a κ-scaled neighbourhood of $f^s(J)$;*
- *$|f^j(J)| < \delta$ for $0 \leq j < s$.*

Then, letting $\psi_0 : J \to I$ and $\psi_s : f^s(J) \to I$ be affine diffeomorphisms, there exists a map $G : I \to I$ in the Epstein class $\mathcal{E}_\beta$ such that $\|\psi_s f^s \psi_0^{-1} - G\|_{C^n} < \varepsilon$. Moreover, $\beta \to \infty$ as $\kappa \to \infty$.

Here, we say that a diffeomorphism $G : I \to I$ is in the *Epstein class* $\mathcal{E}_\beta$ if G^{-1} extends to a (holomorphic) univalent map from $\mathbb{C}_{(-\beta,1+\beta)} := \mathbb{C} \setminus ((-\infty, -\beta] \cup [1 + \beta, \infty))$ into $\mathbb{C}$.

This result, for $n = 2$, appears as part of the proof of the Yoccoz Lemma in [**T**].

Proof of Theorem 1 assuming Theorem 2. By the complex Koebe distortion theorem, the fact that $G \in \mathcal{E}_\beta$ implies that the C^n distance between $G|[0, 1]$ and the identity map is bounded by a constant $\varepsilon(\beta)$, and $\varepsilon(\beta) \to 0$ as $\beta \to \infty$. Taking $\varepsilon = \varepsilon(\beta)$ in Theorem 2, we see that the C^n distance between $\psi_s f^s \psi_0^{-1}|[0, 1]$ and the identity map is at most $2\varepsilon(\beta)$. □

Outline of Proof of Theorem 2. By rescaling the map $f : f^j(J) \to f^{j+1}(J)$, we obtain a diffeomorphism $f_j : I \to I$. For each j, one can find a map $g_j : I \to I$ in the Epstein class such that the C^n distance between f_j and g_j is of order $o(|f^j(J)|)$. Using the classical real Koebe principle (the C^1 version of Theorem 1), we shall prove that $G = g_{s-1} \cdots g_0$ is in the Epstein class $\mathcal{E}_\beta$ (Proposition 5). Finally, using a proposition concerning the composition operator (Proposition 7), we show that $f_{s-1} \cdots f_1$ is C^n close to the map G.

It should be mentioned that similar ideas have appeared in the proofs of Theorem A.6 of [**FM**] and Lemma 3 of [**AMM**], but our result applies in more general situations.

Remark 3. For maps in the class NF^3, the C^1 version of Theorem 1 still holds if we replace the assumption $\sum_{j=0}^{s-1} |f^j(T)| \leq S$ by "$f^s(T)$ is contained in a small neighborhood of critical points which are not in the basin of periodic attractors". See [**K**, **SV**]. It would be interesting to know if the C^k version of Theorems 1 and 2 remain true under this alternative assumption. See also the recent work [**KS**].

2. Proof of Theorem 2

By means of a C^n coordinate change, we may assume that for each critical point c_i, there is a neighborhood U_i of c_i such that $|f(x) - f(c)| = |x - c_i|^{\alpha_i}$ for $x \in U_i$. Let us also fix an open interval $U_i' \ni c_i$ such that $\overline{U_i'} \subset U_i$.

Define $U := \bigcup_i U_i$ and $U' := \bigcup_i U_i'$. Let $\eta = d(\partial U, \partial U')$. Then any interval of length less than η is either contained in U or disjoint from U'.

We fix T, J, κ, S as in Theorem 2. Let $J_0 = J$ and $J_i = f^i(J)$. For every $0 \le i < s$ we have a diffeomorphism $f^{s-i} : f^i(T) \to f^s(T)$ where $f^s(T)$ is a κ-scaled neighbourhood of $f^s(J)$.

We will rescale our maps as follows. Let $\psi_i : J_i \to I$ be the affine homeomorphisms such that each $f_i = \psi_{i+1} f \psi_i^{-1}$ is monotone increasing. Then the following diagram commutes.

$$\begin{tikzcd}
J_0 \arrow[r, "f"] \arrow[d, "\psi_0"'] & J_1 \arrow[r, "f"] \arrow[d, "\psi_1"] & \cdots \arrow[r, "f"] \arrow[d] & J_{s-1} \arrow[r, "f"] \arrow[d, "\psi_{s-1}"] & J_s \arrow[d, "\psi_s"] \\
{[0,1]} \arrow[r, "f_0"] & {[0,1]} \arrow[r, "f_1"] & \cdots \arrow[r, "f_{s-2}"] & {[0,1]} \arrow[r, "f_{s-1}"] & {[0,1]}
\end{tikzcd}$$

We then approximate f_i as follows. For $0 \le i \le s-1$, let

$$g_i(x) = \begin{cases} f_i(x) & \text{if } J_i \subset U, \\ \left(1 - \frac{\xi_i}{2}\right) x + \frac{\xi_i}{2} x^2 & \text{otherwise,} \end{cases}$$

where $\xi_i = \int_0^1 D^2 f_i(t)dt$.

We use $C^n(I)$ to denote Banach space of C^n maps $\phi : I \to \mathbb{R}$ with the C^n-norm

$$\|h\|_n = \max\{|D^k \phi(x)| : 0 \le k \le n, x \in I\}.$$

Let $C^n(I; I)$ denote the space of maps $\phi \in C^n(I)$, with the same norm, such that $\phi(I) \subset I$. Let $\mathrm{Diff}_+^n(I)$ denote the space of all orientation-preserving C^n automorphisms of I.

Lemma 4. *There exists a continuous monotone increasing function* $w : (0, \infty) \to (0, \infty)$ *(depending on* f*) such that* $\lim_{t \to 0+} w(t) = 0$ *and such that for all* $0 \le i \le s-1$,

$$\|g_i - f_i\|_n \le w(|J_i|)|J_i|.$$

Proof. Assume J_i is not in U, otherwise $g_i = f_i$. We will first estimate $|D^2 g_i(x) - D^2 f_i(x)|$ for $x \in [0,1]$. Observe that $D^2 g_i(x) = \xi_i = \int_0^1 D^2 f_i(t)dt$ and $D^2 f_i(x) = \frac{|J_i|^2}{|J_{i+1}|} D^2 f(\psi_i^{-1}(x))$. Since there exists some $x_0 \in [0,1]$ with $\int_0^1 D^2 f_i(t)dt = D^2 f_i(x_0)$, so $D^2 g_i(x) = D^2 f_i(x_0)$ and

$$\begin{aligned} |D^2 g_i(x) - D^2 f_i(x)| &= |D^2 f_i(x_0) - D^2 f_i(x)| \\ &= \frac{|J_i|^2}{|J_{i+1}|} |D^2 f(\psi_i^{-1}(x_0)) - D^2 f(\psi_i^{-1}(x))| \\ &\le \frac{|J_i|^2}{|J_{i+1}|} w_1(|J_i|) \le C|J_i| w_1(|J_i|), \end{aligned}$$

where w_1 is the modulus of continuity of $D^2 f$, i.e. the function $w(\varepsilon) = \sup_{|x-y|<\varepsilon} |D^2 f(x) - D^2 f(y)|$, and $C = \sup_{x \notin U'} |Df(x)|^{-1}$.

Note that there exists some $x_1 \in [0,1]$ such that $Df_i(x_1) = Dg_i(x_1)$. So for $x \in [0,1]$,

$$|Dg_i(x) - Df_i(x)| \le \int_{x_1}^{x} |D^2 g_i(t) - D^2 f_i(t)| dt \le C|J_i| w_1(|J_i|).$$

Similarly,

$$|g_i(x) - f_i(x)| \le \int_0^x |Dg_i(t) - Df_i(t)| dt \le C|J_i| w_1(|J_i|).$$

For any $2 < k \le n$, $D^k g_i = 0$. Hence, for $x \in I$,

$$|D^k(g_i - f_i)(x)| = |D^k f_i(x)| = \frac{|J_i|^k}{|J_{i+1}|} |D^k f(\psi_i^{-1}(x))| \le C|J_i|^{k-1}.$$

Setting $w(t) = C \max(w_1(t), t)$ completes the proof. □

The map $g_{s-1} \cdots g_0$ is our candidate for G. Let us first apply the classical real Koebe principle to prove that G is in the Epstein class.

Proposition 5. *Assume that* $\sup_{j=0}^{s-1} |f^j(J)|$ *is sufficiently small. Then for each* $0 \le j \le s-1$, $g_{s-1} \cdots g_j$ *belongs to the Epstein class* $\mathcal{E}_\beta$, *where* $\beta > 0$ *is a constant depending only on* κ. *Moreover,* $\beta \to \infty$ *as* $\kappa \to \infty$.

Proof. Let T' be the open interval with $J \subset T' \subset T$ such that both components of $f^s(T') \setminus f^s(J)$ have length $\kappa |f^s(J)|/2$. Let $\hat{T}'_j = \psi_j(f^s(T'))$ for all $0 \le j \le s$. Clearly f_j extends to a diffeomorphism from $\hat{T}'_j$ onto $\hat{T}'_{j+1}$. By the classical real Koebe principle, there exists a constant $C = C(\kappa) > 1$ such that provided that $\sup_{j=0}^{s-1} |f^j(T)|$ is small enough, then for all $x, y \in T'$ we have $|Df^s(x)|/|Df^s(y)| \le C$. Therefore, for each $0 \le j \le s-1$, $f_{s-1} \cdots f_j$ is a well-defined diffeomorphism from $\hat{T}'_j$ onto $\hat{T}'_s$ with derivative between $1/C$ and C. Clearly, there exists $\beta = \beta(\kappa) > 0$ such that $\hat{T}'_j \supset [-2\beta, 1+2\beta]$ for all j and moreover $\beta \to \infty$ as $\kappa \to \infty$.

Note that for each $0 \le j \le s-1$, g_j^{-1} extends to a univalent map from $\mathbb{C}_{T'_{j+1}}$ into $\mathbb{C}_{T'_j}$. Moreover, for a given κ, arguing as in the previous lemma, we have that for all $0 \le j \le s-1$,

$$\sup_{y \in \hat{T}'_j} |f_j(y) - g_j(y)| = o(|J_j|).$$

Claim. There exists δ such that if $\sup_{j=0}^{s-1} |f^j(J)| < \delta$ then for any $x \in \hat{T}'_0$ and any $0 \le r \le s-1$, if $g_j \cdots g_0(x) \in \hat{T}'_{j+1}$ for all $0 \le j \le r-1$, then

$$|f_{r-1} \cdots f_0(x) - g_{r-1} \cdots g_0(x)| < \frac{\beta}{2C}.$$

To prove this claim, let $A_r = B_{-1} = id$ and for all $0 \le i \le r-1$ let $A_i = f_{r-1}\cdots f_i$ and $B_i = g_i \cdots g_0$. Then

$$\begin{aligned}
&|f_{r-1}\cdots f_0(x) - g_{r-1}\cdots g_0(x)|\\
&\qquad = |A_0 B_{-1}(x) - A_r B_{r-1}(x)| \le \sum_{i=0}^{r-1} |A_i B_{i-1}(x) - A_{i+1}B_i(x)|\\
&\qquad = \sum_{i=0}^{r-1} |A_{i+1} f_i B_{i-1}(x) - A_{i+1} g_i B_{i-1}(x)|\\
&\qquad \le \sum_{i=0}^{r-1} \sup_{z\in \hat{T}'_{i+1}} |A_{i+1}(z)| \sup_{y\in\hat{T}'_i} |f_i(y) - g_i(y)| \le C\sum_{i=0}^{r-1} o(|J_i|)|J_i|,
\end{aligned}$$

which is arbitrarily small provided that $\sup_{j=0}^{s-1} |f^j(J)|$ is small enough. This proves the claim.

For $x \in I^\beta := [-\beta, 1+\beta]$ and $0 \le r \le s-1$, $d(f_{r-1}\cdots f_0(x), \partial\hat{T}'_r) \ge \beta/C$. Together with the claim, this implies (by induction on r) that for all $0 \le r \le s-1$, $g_{r-1}\cdots g_0$ is well-defined on I^β and maps I^β diffeomorphically onto a subinterval of $\hat{T}'_r$. Since $f_{s-1}\cdots f_0(I^\beta) \supset I^{\beta/C}$, applying the claim once again gives us $G(I^\beta) \supset I^{\beta/2C}$. This proves that for any $0 \le j \le s-1$, $g_j^{-1}\cdots g_{s-1}^{-1}$ extends to a univalent map from $\mathbb{C}_{I^{\beta/2C}}$, so $g_{s-1}\cdots g_j$ is in the Epstein class $\mathcal{E}_{\beta/2C}$. Redefining β completes the proof. □

Applying the complex Koebe distortion theorem, this implies the following.

Corollary 6. *There exists a constant $C = C(\kappa) > 0$ such that for any $0 \le j \le s-1$, we have*

$$\|\log D(g_{s-1}\cdots g_j)\|_n \le C.$$

The proof of Theorem 2 is then completed by the following proposition and lemma.

Proposition 7. *Let $n \in \mathbb{N}\cup\{0\}$, let $g_j \in \mathrm{Diff}^{n+1}_+(I)$, $f_j \in \mathrm{Diff}^n_+(I)$, for $0 \le j \le s-1$. For any $C > 1$ there exists $E = E(C,n) > 0$ such that if the following hold:*

1. *for each $0 \le j < s$, $\|\log D(g_{s-1}\cdots g_j)\|_n \le C$;*
2. *if $n \ge 1$, $\|\log Dg_j - \log Df_j\|_{n-1} \le C$ for all $0 \le j \le s-1$;*
3. *$\sum_{j=0}^{s-1} \|g_j - f_j\|_n \le C$,*

then

$$\|g_{s-1}\cdots g_0 - f_{s-1}\cdots f_0\|_n \le E\sum_{j=0}^{s-1}\|f_j - g_j\|_n.$$

The proof of this proposition will be given in the next section.

Lemma 8. *For any $C > 1$ and $k \in \mathbb{N}$, there exists $C' = C'(C,k) > 1$ with the following property. Let $\phi, \tilde{\phi} \in C^k(I)$ have $\|\phi\|_k, \|\tilde{\phi}\|_k \le C$. Then*

(1) $\|e^{\phi}\|_k \leq C'$;
(2) $\frac{1}{C'}\|\phi - \tilde{\phi}\|_k \leq \|e^{\phi} - e^{\tilde{\phi}}\|_k \leq C'\|\phi - \tilde{\phi}\|_k$.

Proof. Let $\psi = e^{\phi}$ and $\tilde{\psi} = e^{\tilde{\phi}}$. By induction it is easy to compute that for all $k \geq 1$, there exist polynomials P_k and Q_k such that

- $D^k(e^{\phi}) = e^{\phi} \cdot P_k(\phi, D\phi, \ldots, D^k\phi)$;
- $D^k(\phi) = Q_k(\psi, D\psi, \ldots, D^k\psi)/\psi^k$.

From these the lemma follows easily. □

Proof of Theorem 2 assuming Proposition 7. It suffices to check that the conditions in Proposition 7 are satisfied. The first condition was verified in Corollary 6. By Lemma 4, $\|f_j - g_j\|_n \leq |J_j|w(|J_j|)$. Furthermore, from the proof of that lemma, we can show that $\|\log Df_j\|_{n-1}, \|\log Dg_j\|_{n-1}$ are bounded above. Whence by Lemma 8, provided that $\sup_{j=0}^{s-1}|f^j(J)|$ is small enough, the second condition is verified. For the third one, we use the assumption $\sum_{j=0}^{s-1}|f^j(J)| \leq \sum_{j=0}^{s-1}|f^j(T)| \leq S$ and the fact that $w(|J_j|)$ is small when $|J_j|$ is small. □

3. Proof of Proposition 7

The goal of this section is to prove Proposition 7. Let us begin with a small lemma.

Lemma 9. *For any $k \in \mathbb{N} \cup \{0\}$ and $C > 0$ there exists $K = K(C, k)$ with the following property. Let $u, v, B \in C^k(I; I)$, and let $A \in C^{k+1}(I)$. Assume that $\|A\|_{k+1} \leq C$ and $\|B\|_k \leq C$. Then*

$$\|AuB - AvB\|_k \leq K\|u - v\|_k.$$

Proof. This lemma is a straightforward consequence of the chain rule. □

Proof of Proposition 7. We first introduce some notation for our calculations. Let $A_s = B_{-1} = id$ and for $0 \leq j \leq s-1$, let $A_j = g_{s-1} \cdots g_j$ and $B_j = f_j \cdots f_0$. Then

$$\begin{aligned} g_{s-1} \cdots g_0 - f_{s-1} \cdots f_0 &= A_0 B_{-1} - A_s B_{s-1} \\ &= \sum_{j=0}^{s-1}(A_j B_{j-1} - A_{j+1} B_j) \\ &= \sum_{j=0}^{s-1}(A_{j+1} g_j B_{j-1} - A_{j+1} f_j B_{j-1}). \end{aligned}$$

Writing $S_j := A_j B_{j-1} = A_{j+1} g_j B_{j-1} = g_{s-1} \cdots g_j f_{j-1} \cdots f_0$, we have

$$g_{s-1} \cdots g_0 - f_{s-1} \cdots f_0 = \sum_{j=0}^{s-1}(S_j - S_{j+1}).$$

The proof of the proposition will proceed by induction on n. First, by Lemmas 8 and 9, $\|S_j - S_{j+1}\|_0 \le K(C,0)\|f_j - g_j\|_0$. Thus, $\|g_{s-1}\cdots g_0 - f_{s-1}\cdots f_0\|_0 \le \sum_{i=0}^{s-1}\|f_j - g_j\|_0$. This proves the lemma for the case $n = 0$.

Now let $m \ge 1$ and assume that the proposition holds for $n = m-1$. Let us prove it for $n = m$.

First, for each $0 \le r \le s-1$, applying the induction hypothesis to the mappings f_j, g_j, $0 \le j \le r$, we have

$$\|f_r\cdots f_0 - g_r\cdots g_0\|_{m-1} \le E_1 \sum_{i=0}^{j-1}\|f_i - g_i\|_{m-1}, \tag{1}$$

where E_1 is a constant (depending only on C and m). Also, it is easy to show that the first assumption of the proposition implies $\|\log D(g_r\ldots g_0)\|_n < 2C$. Therefore, by the first part of Lemma 8 we have $\|D(g_r\ldots g_0)\|_n < C'$. Hence,

$$\|g_r\cdots g_0\|_m = \max(1, \|D(g_r\cdots g_0)\|_{m-1}) \le C'.$$

Applying this to (1), we have

$$\|B_r\|_{m-1} \le C_1. \tag{2}$$

To complete the induction it suffices to prove that there exists a constant E_2 such that

$$\|D^m(S_j - S_{j+1})\|_0 \le E_2\|f_j - g_j\|_m. \tag{3}$$

To this end let us first prove the following.

Claim. There exists a constant C_2 depending only on C such that for all $0 \le j \le s$, $\|\log DS_j - \log DS_{j+1}\|_{m-1} \le C_2\|f_j - g_j\|_m$.

In fact, for each $0 \le j \le s-1$, by the chain rule,

$$\begin{aligned}
&\log DS_j - \log DS_{j+1}\\
&\quad = [\log(DA_{j+1}g_jB_{j-1}) + \log(Dg_jB_{j-1}) + \log DB_{j-1}]\\
&\qquad\qquad - [\log(DA_{j+1}f_jB_{j-1}) + \log(Df_jB_{j-1}) + \log DB_{j-1}]\\
&\quad = [\log(DA_{j+1}g_jB_{j-1}) - \log(DA_{j+1}f_jB_{j-1})]\\
&\qquad\qquad + [\log(Dg_jB_{j-1}) - \log(f_jB_{j-1})]\\
&\quad =: P_j + Q_j.
\end{aligned}$$

From the assumption $\|\log DA_{j+1}\|_m \le C$ and from (2), by Lemma 9, we obtain

$$\|P_j\|_{m-1} \le K(C_1, m-1)\|f_j - g_j\|_{m-1},$$

and

$$\|Q_j\|_{m-1} \le K(C_1, m-1)\|\log Dg_j - \log Df_j\|_{m-1}.$$

Since $\|\log Dg_j\|_{m-1}$ and $\|\log Df_j\|_{m-1}$ are bounded from above, the second statement of Lemma 8 implies the claim.

Finally let us deduce (3) from the claim. By the second part of Lemma 8, it suffices to show that $\|\log DS_j\|_{m-1}$ is bounded from above by a constant.

Since $\|\log DS_0\|_{m-1} = \|\log DA_0\|_{m-1} \leq C$, this follows from the third assumption by applying the claim. This completes the proof.

□

References

[**AMM**] A. Avila, M. Martens, W. de Melo *On the dynamics of the renormalisation operator,* preprint, 2000.

[**FM**] E. de Faria, W. de Melo *Rigidity of critical circle mappings I,* J. Eur. Math. Soc. **1** (1999), 339-392.

[**K**] O. Kozlovski, *Getting rid of the negative Schwarzian derivative condition*, Ann. Math. (2), **152** (2000), no. 3, 743-762.

[**KS**] O. Kozlovski and D. Sands *The real koebe lemma*, Manuscript.

[**MS**] W. de Melo and S. van Strien, *One dimensional dynamics,* Ergebnisse Series **25**, Springer-Verlag, 1993.

[**SV**] S. van Strien, E. Vargas, *Real Bounds, ergodicity and negative Schwarzian for multimodal maps,* preprint, 2002.

[**Su**] D. Sullivan, *Bounds, quadratic differentials, and renormalisation conjectures,* AMS Centennial Publications, vol. 2, Mathematics into the Twenty-first Century, 1992.

[**T**] M. Todd, *One dimensional dynamics: cross-ratios, negative Schwarzian and structural stability,* thesis, University of Warwick, 2003.

Mathematics Institute, University of Warwick,
Coventry, CV4 7AL

E-mail address: wxshen@maths.warwick.ac.uk

Mathematics Department, University of Surrey, Guildford,
Surrey, GU2 7XH, UK

E-mail address: m.todd@surrey.ac.uk