# TRANSIENCE IN DYNAMICAL SYSTEMS

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Abstract. We extend the theory of transience to general dynamical systems with no Markov structure assumed. This is linked to the theory of phase transitions. We also provide new examples to illustrate different kinds of transient behaviour.

## 1. Introduction

A classical question in the theory of random walks, Markov chains and ergodic theory is whether a system is recurrent or transient. The notion of recurrence and its consequences are well understood in the former two cases [Fe, V1, V2]. Whereas in the realm of ergodic theory it is not even clear what good definitions of recurrence or transience are. The definition involves a triple $(X, f, \varphi)$, where $f : X \to X$ is a dynamical system and $\varphi : X \to \mathbb{R}$ is a function (or potential). In the context of countable Markov shifts, with a fairly weak assumption on the smoothness of the potential, Sarig [S2] gave a definition of recurrence which comes naturally from the theory of Markov chains. This definition even applies in situations such as those studied in [H, MP, HK, Lo], which can be thought of as non-uniformly hyperbolic, although the dynamical system still has a well-defined Markov structure. The aim of this paper is to provide a definition of transience which applies in a much more general context. In particular, the definition we propose (see Definition 2.5) requires no Markov structure for $(X, f)$, therefore we can not make use of the corresponding notions for random walks or Markov chains. We also only need very weak assumptions on the smoothness of potentials.

In Section 5 we show that our definition can be checked even for certain non-uniformly hyperbolic dynamical systems with no canonical Markov structure. We particularly focus on multimodal interval maps with the so-called 'geometric potentials', since this class of systems combines both a highly non-Markov structure and quite singular potentials. This analysis passes to many other classes of interval maps.

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Our definition of recurrence/transience makes use of thermodynamic formalism. Given a system $(X, f, \varphi)$, one can study the recurrence/transience within the family of potentials $\{t\varphi : t \in \mathbb{R}\}$. In the cases we know, the transition at a point $t = t_0$ of $t\varphi$ from recurrence to transience coincides with the pressure function $t \mapsto P(t\varphi)$ not being real analytic at $t_0$. This lack of real analyticity is referred to as a *phase transition* at $t_0$. In Section 6 we examine the transition from recurrence to transience in certain systems, giving some explicit and fairly elementary examples to illustrate the range of possible forms these transitions may take.

## 2. Definition of transience

Our definition of recurrence/transience uses the ideas of pressure as well as conformality and conservativity of measures for a given dynamical system. We first introduce pressure. Throughout we will be dealing with metric spaces $X$, and Borel functions $f : X \to X$, our *dynamical systems*. We define $\mathcal{M}_f$ to be the set of $f$-invariant Borel probability measures. Given a Borel function $\varphi : X \to \mathbb{R}$ (the *potential*), we consider the *pressure* to be

$$P(\varphi) := \sup\left\{h(\mu) + \int \varphi \, d\mu : \mu \in \mathcal{M}_f \text{ and } -\int \varphi \, d\mu < \infty\right\},$$

where $h(\mu)$ denotes the entropy of the measure $\mu$. A measure $\nu \in \mathcal{M}_f$ attaining the above supremum is called an *equilibrium measure/state* for $(X, f, \varphi)$. For many systems, the existence of an equilibrium state is a sufficient condition for recurrence. However, as explained below, there are many examples of systems $(X, f, \varphi)$ with an equilibrium state, but which should not be considered recurrent.

The main tool used to study pressure, equilibrium states, and, as we will see, recurrence, is the *Transfer (or Ruelle) operator*, which is defined by

$$(L_\varphi g)(x) = \sum_{Ty=x} g(y) \exp(\varphi(y)) \tag{1}$$

for $x \in X$, $\varphi$ and $g$ in some suitable Banach spaces (constructing suitable Banach spaces where this operator acts and behaves well is an important line of research in the field). As we will see in the classical result Theorem 3.1, in good cases, there exists $\lambda > 0$, a finite Borel probability measure $m$ on $X$ and a continuous function $h : X \to [0, \infty)$ such that

$$L_\varphi h = \lambda h \text{ and } L_\varphi^* m = \lambda m, \tag{2}$$

where $\lambda = e^{P(\varphi)}$ and $L_\varphi^*$ is the dual operator of $L_\varphi$ (i.e, for a continuous function $\psi : X \to \mathbb{R}$, then $\int \psi \, dm = \int L_\varphi \psi \, dm$). If, moreover, $\int h \, dm < \infty$ then the measure $\mu := \frac{hm}{\int h \, dm}$ is in $\mathcal{M}_f$. Such measures $m$ will occupy an important place in this work and are the subject of the following definition.

**Definition 2.1.** *Given $f : X \to X$, a dynamical system and $\varphi : X \to \mathbb{R}$ a potential, we say that a Borel measure $m$ on $X$ is* $\varphi$-conformal *if*

$$L_\varphi^* m = m$$

*and there exists an at most countable collection of open sets $\{U_i\}_i \subset X$ such that $m(X \setminus \cup_i U_i) = 0$ and $m(U_i) < \infty$ for all $i$.*

**Remark 2.1.** *Notice that by this definition, $m$ in (2) is $(\varphi - \log\lambda)$-conformal. Also we point out that some authors would consider such $m$ to be $\varphi$-conformal; unless $\lambda = 1$, we do not.*

*Also note that if $f^n : U \to f^n(U)$ is 1-to-1 then $m(f^n(U)) = \int_U e^{-S_n\varphi}\, dm$ where*

$$S_n\varphi(x) := \varphi(x) + \cdots + \varphi \circ f^{n-1}(x).$$

*A dynamical system is* topologically exact*, if for any open set $U \subset X$ there exists $n \in \mathbb{N}$ such that $f^n(U) = X$. If our system is topologically exact and $\varphi$ is a bounded potential, then by the above $m(X) < \infty$. Whenever the measure of $X$ is finite, we normalise.*

We will define $(X, f, \varphi)$ to be recurrent if there is a $(\varphi - P(\varphi))$-conformal measure which has the properties outlined below.

A measure $\mu$ on $X$ is called *$f$-non-singular* if $\mu(A) = 0$ if and only if $\mu(f^{-1}(A)) = 0$. A set $W \subset X$ is called *wandering* if the sets $\{f^{-n}(W)\}_{n=0}^{\infty}$ are disjoint.

**Definition 2.2.** *Let $f : X \to X$ be a dynamical system. An $f$-non-singular measure $\mu$ is called* conservative *if every wandering set $W$ is such that $\mu(W) = 0$.*

A conservative measure satisfies the Poincaré Recurrence Theorem (see [Aa, p.17], or [S4, p.31]).

A further property we would like any set of recurrent points to satisfy is given in the following definition.

**Definition 2.3.** *Let $B_\varepsilon(x_0)$ denote the open ball of radius $\varepsilon$ centred at the point $x_0$. We say that $x \in X$* goes to $\varepsilon$-large scale *at time $n$ if there exists an open set $U \ni x$ such that $f^n : U \to B_\varepsilon(f^n(x))$ is a bijection. We say that $x$* goes to $\varepsilon$-large scale infinitely often *if there exists $\varepsilon > 0$ such that $x$ goes to $\varepsilon$-large scale for infinitely many times $n \in \mathbb{N}$. Let $LS_\varepsilon \subset X$ denote the set of points which go to $\varepsilon$-large scale infinitely often.*

**Definition 2.4.** *Given a Borel measure $\mu$ on $X$ we say that $\mu$ is* weakly expanding *if there exists $\varepsilon > 0$ such that $\mu(LS_\varepsilon) > 0$.*

We use the term 'weakly expanding' for our measures to distinguish from the expanding measures in [Pi] (note that those measures go to large scale with *positive frequency*). As we note in Remark 3.1, for a dynamical system with a Markov structure, any conservative measure is automatically weakly expanding. We are now ready to define recurrence.

**Definition 2.5.** *Let $f : X \to X$ be a dynamical system and $\varphi : X \to [-\infty, \infty]$ a Borel potential. Then $(X, f, \varphi)$ is called* recurrent *if there is a finite weakly expanding conservative $(\varphi - P(\varphi))$-conformal measure $m$. Moreover, if there exists a finite $f$-invariant measure $\mu \ll m$, then we say that $\varphi$ is* positive recurrent*; otherwise we say that $\varphi$ is* null recurrent*.*

*If the system $(X, f, \varphi)$ is not recurrent it is called* transient*.*

Sarig defined recurrence in the setting of countable Markov shifts with fairly well-behaved potentials, as in (3) below. We will compare these two definitions of recurrence: in particular we give examples, outside the context used by Sarig, which the definition in (3) couldn't handle and show that ours can handle them. An easily stated such example is the following.

**Example 2.1.** *Let $\beta > 1$ be a real number. The $\beta$-transformation is the interval map $T_\beta : [0,1) \to [0,1)$ defined by $T_\beta(x) = \beta x \mod 1$. This map is Markov for only countably many values of $\beta$. It was shown by Walters* [W2] *that for any $\beta > 1$, if $\varphi : [0,1) \to \mathbb{R}$ is a Lipschitz potential then there exists a finite weakly expanding conservative $(\varphi - P(\varphi))$-conformal measure $m$. Moreover, there exists a finite $T_\beta$-invariant measure $\mu \ll m$. That is, if $\varphi$ is a Lipschitz potential then the triple $([0,1), T_\beta, \varphi)$ is positive recurrent.*

We will be particularly interested in systems $(X, f, \varphi)$ where for some values of $t$, $(X, f, t\varphi)$ is recurrent, and for others it is transient. This phenomenon is associated to the smoothness of the pressure function $p_\varphi(t) := P(t\varphi)$. If the function is not real analytic at some $t_0 \in \mathbb{R}$ then we say there is a *phase transition* at $t_0$. If this function is not even $C^1$ at $t_0$ then we say that there is a *first order phase transition* at $t_0$. Phase transitions can also be associated with the non-uniqueness of equilibrium states, but this is not always the case, as shown in Section 4.1 (see case (iv) in Figure 1). For our examples, we could alternatively characterise the point $t_0$ as: for any open interval $U \ni t_0$ there exist $t_1, t_2 \in U$ such that system $(X, f, t_1\varphi)$ is recurrent, and $(X, f, t_1\varphi)$ is transient. We will give further examples to show what can happen at phase transitions.

We finish this section by asking some questions our definition of transience raises:

- Are there examples for which our definition of recurrence/transience conflicts with that in (3)? In all of the examples we have here, if the definition given by (3) is well-defined, then so is ours and they coincide.
- Is our definition of transience really more widely applicable than that given by (3)? Example 2.1 already gives evidence that this is so. We give further evidence in Section 4.
- We tend to see transience kick in after/before some phase transition, so if this were always the case then we could define transience in this way. Is it true that given a system $(X, f, \varphi)$ where $(X, f, t\varphi)$ is recurrent for *all* $t < t_0$, and there is a phase transition at $t_0$ then the system is transient for all $t > t_0$? We construct examples where this is not the case in Section 6, see also [S3, Example 3].
- What does the existence of a dissipative $(t_0\varphi - p_\varphi(t_0))$-conformal measure tell us about a phase transition at $t_0$?

The structure of the paper is as follows

- In Section 3, the theory of recurrence and transience is presented in the best understood setting (primarily through the work of Cyr and Sarig), the Markov shift case, firstly in the finite alphabet case, and then in the countable. We show how our definition of recurrence fits in with this case, give a very brief sketch of some examples, as well as discussing an alternative type of definition of transience due to Cyr and Sarig.

- In Section 4, examples of interval maps which exhibit both transient and recurrent behaviour are given. The transient behaviour is due to either the lack of smoothness of the potentials or the lack of hyperbolicity of the underlying dynamical system. Our definition of transience is shown to be well-suited to this setting. Sarig's definition of recurrence/transience is not tractable in many of these cases. We also discuss the transition from recurrence to transience here. Our primary applications are to multimodal interval maps $f : I \to I$ with the geometric potential $-\log|f'|$, which are discussed in detail in Section 5.
- In Section 6, we give a class of simple interval maps and fairly elementary potentials which exhibit a range of different behaviours at the transition from recurrence to transience.

## 3. Symbolic spaces

In this section we discuss thermodynamic formalism in the context of Markov shifts. We review some results concerning the existence and uniqueness of equilibrium measures. We also discuss the regularity properties of the pressure function. Many properties of Markov shifts defined in finite alphabets are different to those for a countable alphabet. The lack of compactness of the latter shifts is a major obstruction for the existence of equilibrium measures and can also result in transience. Symbolic spaces are of particular importance, not only because of their intrinsic interest, but also because they provide models for uniformly and non-uniformly hyperbolic dynamical systems (see for example [Bo1, Ra, Lo, S5]).

Let $S \subset \mathbb{N}$ be the *alphabet* and $\mathcal{T}$ be a matrix $(t_{ij})_{S\times S}$ of zeros and ones (with no row and no column made entirely of zeros). The corresponding *symbolic space* is defined by

$$\Sigma := \left\{x \in S^{\mathbb{N}_0} : t_{x_i x_{i+1}} = 1 \text{ for every } i \in \mathbb{N}_0\right\},$$

and the shift map is defined by $\sigma(x_0x_1\cdots) = (x_1x_2\cdots)$. If the alphabet $S$ is finite we say that $(\Sigma, \sigma)$ is a *finite Markov shift*, if $S$ is (infinite) countable we say that $(\Sigma, \sigma)$ is a *countable Markov shift.* Given $n \geqslant 0$, the *word* $x_0\cdots x_{n-1} \in S^n$ is called *admissible* if $t_{x_i x_{i+1}} = 1$ for every $0 \leqslant i \leqslant n-2$. We will always assume that $(\Sigma, \sigma)$ is *topologically mixing*, except in Section 3.3 where the consequences of not having this hypothesis are discussed. This is equivalent to the following property: for each pair $a, b \in S$ there exists $N \in \mathbb{N}$ such that for every $n > N$ there is an admissible word $\underline{a} = a_0 \ldots a_{n-1}$ of length $n$ such that $a_0 = a$ and $a_{n-1} = b$. If the alphabet $S$ is finite this is also equivalent to the existence of an integer $N \in \mathbb{N}$ such that every entry of the matrix $\mathcal{T}^N$ is positive.

We equip $\Sigma$ with the topology generated by the $n$-cylinder sets:

$$C_{i_0\cdots i_{n-1}} := \{x \in \Sigma : x_j = i_j \text{ for } 0 \leqslant j \leqslant n-1\}.$$

We let $C(\Sigma)$ be the set of continuous functions $\varphi : I \to \mathbb{R}$. Given a function $\varphi\colon \Sigma \to \mathbb{R}$, for each $n \geqslant 1$ we set

$$V_n(\varphi) := \sup\left\{|\varphi(x) - \varphi(y)| : x, y \in \Sigma,\ x_i = y_i \text{ for } 0 \leqslant i \leqslant n-1\right\}.$$

Note that $\varphi : \Sigma \to \mathbb{R}$ is continuous if and only if $V_n(\varphi) \to 0$. The regularity of the potentials that we consider is fundamental when it comes to proving existence of equilibrium measures as well as recurrence/transience.

**Definition 3.1.** *We say that* $\varphi : \Sigma \to \mathbb{R}$ *has* summable variations *if* $\sum_{n=2}^{\infty} V_n(\varphi) < \infty$. *Clearly, if* $\varphi$ *has summable variations then it is continuous. We say that* $\varphi : \Sigma \to \mathbb{R}$ *is* weakly Hölder continuous *if* $V_n(\varphi)$ *decays exponentially, that is there exists* $C > 0$ *and* $\theta \in (0, 1)$ *such that* $V_n(\varphi) < C\theta^n$ *for all* $n \geqslant 2$. *If this is the case then clearly it has summable variations.*

Note that in this symbolic context, given any symbolic metric, the notions of Hölder and Lipschitz function are essentially the same (see [PP, p.16]).

We say that $\mu$ is a *Gibbs measure* on $\Sigma$ if there exist $K, P \in \mathbb{R}$ such that for every $n \geqslant 1$, given an $n$-cylinder $C_{i_0 \cdots i_{n-1}}$,

$$\frac{1}{K} \leqslant \frac{\mu(C_{i_0 \cdots i_{n-1}})}{e^{S_n\varphi(x) - nP}} \leqslant K$$

for any $x \in C_{i_0 \cdots i_{n-1}}$. We will usually have $P = P(\varphi)$.

**Remark 3.1.** *In the topologically mixing Markov shift* $(\Sigma, \sigma)$ *case, due to the Markov structure, and since for any conservative measure* $m$ *satisfies the Poincaré Recurrence Theorem,* $m$*-a.e. point goes to large scale infinitely often, even in the countable Markov shift case. Hence in our definition of transience, we can drop the weakly expanding requirement.*

### 3.1. **Compact case.**

When the alphabet $S$ is finite, the space $\Sigma$ is compact. Moreover, the entropy map $\mu \mapsto h(\mu)$ is upper semi-continuous. Therefore, continuous potentials have equilibrium measures. In order to prove uniqueness of such measures, regularity assumptions on the potential and a transitivity/mixing assumption on the system are required. The following is the Ruelle-Perron-Frobenius Theorem (see [Bo3, p.9] and [PP, Proposition 4.7]).

**Theorem 3.1.** *Let* $(\Sigma, \sigma)$ *be a topologically mixing finite Markov shift and let* $\varphi : \Sigma \to \mathbb{R}$ *be a Hölder potential. Then*

*(a) there exists a* $(\varphi - P(\varphi))$*-conformal measure* $m_\varphi$*;*
*(b) there exists a unique equilibrium measure* $\mu_\varphi$ *for* $\varphi$*;*
*(c) there exists a positive function* $h_\varphi \in L^1(m_\varphi)$ *such that* $L_\varphi h_\varphi = e^{P(\varphi)} h_\varphi$ *and* $\mu_\varphi = h_\varphi m_\varphi$*;*
*(d) For every* $\psi \in C(\Sigma)$ *we have*

$$\lim_{n \to \infty} \left\| e^{-nP(\varphi)} L_\varphi^n(\psi) - \left( \int \psi \, d\mu_\varphi \right) h \right\|_\infty = 0.$$

*(e)* $m_\varphi$ *and* $\mu_\varphi$ *are Gibbs measures;*
*(f) the pressure function* $t \mapsto P(t\varphi)$ *is real analytic on* $\mathbb{R}$*.*

Part (c) of this theorem implies that these systems are positive recurrent according to our definition (this is also the case according to Sarig's definition). Notice also that part (f) of this theorem implies that such systems have no phase transitions.On the other hand, Hofbauer [H] showed that for a particular class of non-Hölder potentials $\varphi$, there are phase transitions, and also equilibrium states need not be unique. We describe this example in Section 4.1. We are led to the following natural question:

**Question:** how much can we relax the regularity assumption on the potential and still have uniqueness of the equilibrium measure/recurrence?

In order to give a partial answer to this question, Walters [W3] introduced the following class of functions.

**Definition 3.2.** *For $\varphi : \Sigma \to \mathbb{R}$, we say that $\varphi : \Sigma \to \mathbb{R}$ is a* Walters function *if for every $p \in \mathbb{N}$ we have $\sup_{n\geqslant 1} V_{n+p}(S_n\varphi) < \infty$ and*

$$\lim_{p\to\infty} \sup_{n\geqslant 1} V_{n+p}(S_n\varphi) = 0.$$

*We say that $\varphi : \Sigma \mapsto \mathbb{R}$ is a* Bowen function *if*

$$\sup_{n\geqslant 1} V_n(S_n\varphi) < \infty.$$

Note that if $\varphi$ is of summable variations then it is a Walters function. Walters showed that if a potential $\varphi$ is Walters then it satisfies the Ruelle-Perron-Frobenius Theorem. In particular it has a unique equilibrium measure. Bowen introduced the class of functions we call Bowen in [Bo2]. Note that every Walters function is a Bowen function and that there exist Bowen functions which are not Walters [W5]. Bowen functions satisfy conditions (a)-(e) of Theorem 3.1, but not necessarily (f). The following result was proved by Bowen [Bo1] and Walters [W4] and shows that for $(\Sigma, \sigma)$ a finite Markov shift and $\varphi$ is a Bowen function, the system is recurrent.

**Theorem 3.2** (Bowen-Walters)**.** *For $(\Sigma, \sigma)$ a finite Markov shift, if $\varphi : \Sigma \to \mathbb{R}$ is a Bowen function then there exists a unique equilibrium measure $\mu$ for $\varphi$. Moreover, there exists a conservative $(\varphi - P(\varphi))$-conformal measure and the measure $\mu$ is exact.*

Bowen [Bo1] showed the existence of a unique equilibrium measure and Walters [W4] described the convergence properties of the Ruelle operator of a Bowen function. Recently, Walters [W5] defined a new class of functions that he called 'Ruelle functions' which includes potentials having more than one equilibrium measure.

3.2. **Non-compact case.** The definition of pressure in the case that the alphabet $S$ is finite (compact case) was introduced by Ruelle [Ru1]. In the (non-compact) case when the alphabet $S$ is infinite the situation is more complicated because the definition of pressure using $(n, \epsilon)-$ separated sets depends upon the metric and can be different even for two equivalent metrics. Mauldin and Urbański [MU1] gave a definition of pressure for shifts on countable alphabets satisfying certain combinatorial assumptions. Later, Sarig [S1], generalising previous work by Gurevich [Gu2, Gu1], gave a definition of pressure that satisfies the Variational Principle for any topologically mixing countable Markov shift. This definition and the one given by Mauldin and Urbański coincide for systems where both are defined.

Let $(\Sigma, \sigma)$ be a topologically mixing *countable* Markov shift. This is a non-compact space. Fix a symbol $i_0$ in the alphabet $S$ and let $\varphi \colon \Sigma \to \mathbb{R}$ be a Walters potential.

We refer to the corresponding cylinder $C_{i_0}$ as the *base set* and let

$$Z_n(\varphi, C_{i_0}) := \sum_{x:\sigma^n x = x} e^{S_n\varphi(x)} \mathbb{1}_{C_{i_0}}(x),$$

where $\mathbb{1}_{C_{i_0}}$ is the characteristic function of the cylinder $C_{i_0} \subset \Sigma$. Also, defining

$$r_{C_{i_0}}(x) := \mathbb{1}_{C_{i_0}}(x) \inf\{n \geqslant 1 : \sigma^n x \in C_{i_0}\},$$

we let

$$Z_n^*(\varphi, C_{i_0}) := \sum_{x:\sigma^n x = x} e^{S_n\varphi(x)} \mathbb{1}_{\{r_{C_{i_0}}\}}(x),$$

The so-called *Gurevich pressure* of $\varphi$ is defined by

$$P_G(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log Z_n(\varphi, C_{i_0}).$$

This limit is proved to exist by Sarig [S1, Theorem 1]. Since $(\Sigma, \sigma)$ is topologically mixing, one can show that $P(\varphi)$ does not depend on the base set. This notion of pressure coincides with the usual definition of pressure when the alphabet $S$ is finite and also satisfies the Variational Principle (see [S1]), i.e.,

$$P_G(\varphi) = P(\varphi).$$

Sarig showed in [S2, Theorem 1] that exactly three different kinds of behaviour are possible for a Walters potential[1] $\varphi$ of finite Gurevich pressure. We adopt his definitions of transience and recurrence for a moment:

I. The potential $\varphi$ is *recurrent* if

$$\sum_{n\geqslant 1} e^{-nP(\varphi)} Z_n(\varphi, C_{i_0}) = \infty. \tag{3}$$

Here there exists a conservative $(\varphi - P(\varphi))$-conformal measure $m$. If, moreover
   (a) $\sum_{n\geqslant 1} ne^{-nP(\varphi)} Z_n^*(\varphi, C_{i_0}) < \infty$ then there exists an equilibrium measure for $(\Sigma, \sigma, \varphi)$ absolutely continuous with respect to $m$. This is the *positive recurrent* case;
   (b) $\sum_{n\geqslant 1} ne^{-nP(\varphi)} Z_n^*(\varphi, C_{i_0}) = \infty$ then there is no finite equilibrium measure absolutely continuous with respect to $m$. This is the *null recurrent* case;

II. The potential $\varphi$ is *transient* if

$$\sum_{n\geqslant 1} e^{-nP(\varphi)} Z_n(\varphi, C_{i_0}) < \infty.$$

In this case there is no conservative $(\varphi - P(\varphi))$-conformal measure.

Cases I(a) and I(b) fit our definition (Definition 2.5) of positive and null recurrence respectively, and Case II fits our definition of transience.

[1]Actually, he considered potentials of summable variations but the proofs of his results need no changes if it is assumed that the potential is a Walters function, see [S4].

**Remark 3.2.** *If a potential $\varphi$ is transient then it either has no conformal measure or a dissipative conformal measure. Examples of both cases have been constructed by Cyr* [C1, Section 5]*. Moreover, examples are also given where there is more than one $\varphi$-conformal measure in the transient setting.*

Recently Cyr and Sarig [CS] gave a characterisation of transient potentials which involves a phase transition of some pressure function, indeed they proved:

**Proposition 3.1** (Cyr and Sarig)**.** *The potential $\varphi : \Sigma \to \mathbb{R}$ is transient if and only if for each $i \in S$ there exists $t_0 \in \mathbb{R}$ such that $P(\varphi + t1_{C_i}) = P(\varphi)$ for every $t \leqslant t_0$ and $P(\varphi + t1_{C_i}) > P(\varphi)$ for $t > t_0$.*

Moreover, Cyr [C2] proved that, in a precise sense, most countable Markov shifts have at least one transient potential. We could, in principle, use this as our definition of transience, but outside the domain of Markov shifts it has the disadvantages that it is unclear what kind of set should replace $C_a$, and it would appear to be very hard to check in any case. Also the link between the conditions in Proposition 3.1 and measures has not been established outside the Markov shift setting, which makes it difficult to interpret the condition in an ergodic theory context.

We conclude this section with a very important example of a countable Markov shift, the so called *renewal shift*. Let $S = \{0, 1, 2, \dots\}$ be a countable alphabet. Consider the transition matrix $A = (a_{ij})_{i,j \in S}$ with $a_{0,0} = a_{0,n} = a_{n,n-1} = 1$ for each $n \geqslant 1$ and with all other entries equal to zero. The *renewal shift* is the (countable) Markov shift $(\Sigma_R, \sigma)$ defined by the transition matrix $A$, that is, the shift map $\sigma$ on the space

$$\Sigma_R := \left\{ (x_i)_{i \geqslant 0} : x_i \in S \text{ and } a_{x_i x_{i+1}} = 1 \text{ for each } i \geqslant 0 \right\}.$$

The *induced system* $(\Sigma_I, \sigma)$ is defined as the full-shift on the new alphabet given by $\{C_{0n(n-1)(n-2)\cdots 1} : n \geqslant 1\}$. Given a function $\varphi \colon \Sigma_R \to \mathbb{R}$ with summable variation we define a new function, the *induced potential*, $\Phi \colon \Sigma_R \to \mathbb{R}$ by

$$\Phi(x) := \sum_{k=0}^{r_{C_0}(x)-1} \varphi(\sigma^k x), \tag{4}$$

where the first return map $r_{C_0}$ is defined as above. Sarig [S3] proved that if $\varphi : \Sigma_R \to \mathbb{R}$ is a potential of summable variations, bounded above, with finite pressure and such that the induced potential $\Phi$ is weakly Hölder continuous then there exists $t_c > 0$ such that

$$P(t\varphi) = \begin{cases} \text{strictly convex and real analytic} & \text{if } t \in [0, t_c), \\ At & \text{if } t > t_c, \end{cases}$$

where $A = \sup\{\int \varphi \, d\mu : \mu \in \mathcal{M}\}$. This result is important since several of the examples known to exhibit phase transitions can be modelled by the renewal shift. Indeed, this is the case for the interval examples discussed in Sections 4.1–4.2.

3.3. **Non topologically mixing systems.** All the results we have discussed so far are under the assumption that the systems are topologically mixing. This is a standard irreducibility hypothesis. As we show below, it is easy to construct counterexamples to all the previous theorems when there is no mixing assumption.

Consider the dynamical system $(\Sigma_{0,1} \sqcup \Sigma_{2,3}, \sigma)$, where $\Sigma_{i,j}$ is the full-shift on the alphabet $\{i, j\}$. It is easy to see that the topological entropy of this system is equal to $\log 2$. Moreover, there exist two invariant measures of maximal entropy: the $(1/2, 1/2)$-Bernoulli measure supported in $\Sigma_{0,1}$ and the $(1/2, 1/2)$-Bernoulli measure supported in $\Sigma_{2,3}$. Therefore, the constant (and hence Hölder) potential $\varphi(x) = 0$ has two equilibrium measures. Actually, it is possible to construct a locally constant potential exhibiting phase transitions. Let

$$\psi(x) = \begin{cases} -1 & \text{if } x \in \Sigma_{0,1}, \\ -2 & \text{if } x \in \Sigma_{2,3}. \end{cases}$$

The pressure function has the following form

$$p_\psi(t) = \begin{cases} -t + \log 2 & \text{if } t \geqslant 0; \\ -2t + \log 2 & \text{if } t < 0. \end{cases}$$

Therefore the pressure exhibits a phase transition at $t = 0$. For $t > 0$ the equilibrium state for $t\psi$ is the $(1/2, 1/2)$-Bernoulli measure supported on $\Sigma_{0,1}$ and for $t < 0$ the equilibrium state for $t\psi$ is the $(1/2, 1/2)$-Bernoulli measure supported on $\Sigma_{2,3}$. For $t = 0$ these measures are both equilibrium states. Note that in both cases these measures are also $(t\psi - p_\psi(t))$-conformal, so the phase transitions here are not linked to transience.

Phase transitions caused by the non mixing structure of the system also appear in the case of interval maps. Indeed, the renormalisable examples studied by Dobbs [D] are examples of this type.

## 4. The interval case

In this section we describe examples of systems of interval maps and potentials with phase transitions and explain how our definition of recurrence/transience is an improvement on alternative notions there.

The situation in the compact interval context is different from that of the compact symbolic case in that rather smooth potentials can have more than one equilibrium measure. All the examples we consider are such that entropy map is upper semi-continuous. Since the interval is compact, weak* compactness of the space of invariant probability measures implies that every continuous potential has (at least) one equilibrium measure. The study of phase transitions in the context of topologically mixing interval maps is far less developed that in the case of Markov shifts. We review some of these examples.

4.1. **Hofbauer-Keller.** The following example was constructed by Hofbauer and Keller [HK] based on previous work in the symbolic setting by Hofbauer [H]. We will present it defined in a half open interval, but one can also think of this as a dynamical system on a compact set, namely the circle.

The dynamical system considered is the angle doubling map $f : [0,1) \mapsto [0,1)$ defined by $f(x) = 2x \pmod 1$. Typically this map is studied via its relation to the full shift on two symbols, so the continuity of potentials is only required for the symbolic version of the potential. Given a sequence of real numbers $(a_k)_{k\in\mathbb{N}_0}$ such that $\lim_{k\to\infty} a_k = 0$, we define the potential $\varphi$ by

$$\varphi(x) = \begin{cases} a_k & \text{if } x \in [2^{-k-1}, 2^{-k}), \\ 0 & \text{if } x = 0. \end{cases}$$

Notice that this potential is continuous on $[0,1)$ equipped with the metric induced by the standard metric on the full shift on two symbols.

Let $F$ be the first return map to $X = [1/2, 1)$ with return time $\tau$. So for $X_n := \{\tau\}$, the induced potential $\Phi$ (see (4)) takes the value

$$s_n := \sum_{k=0}^{n-1} a_k.$$

Figure 1 summarises the possible behaviours of the thermodynamic formalism depending on the sums $s_n$. Note that there was mistake[2] in a similar table in the original paper [H], corrected by Walters in [W4, p.1329]. In the final column we apply our definition of recurrence/transience. The first four entries in that column follow directly from results of [H] while the final entry follows from Lemma 4.1 below. The numbering (i)-(v) will be used to refer to the cases in the corresponding row in discussions below.

| | | Pressure $P(\varphi)$ | $\mu_\varphi$ a Gibbs measure? | Unique equiibrium state? | $\varphi$ is +ve recurrent/ transient | |
|---|---|---|---|---|---|---|
| $\sum_n e^{s_n} > 1$ | $\sum_k a_k$ converges | $P(\varphi) > 0$ | yes | yes | +ve recurrent | (i) |
| | $\sum_k a_k$ diverges | $P(\varphi) > 0$ | no | yes | +ve recurrent | (ii) |
| $\sum_n e^{s_n} = 1$ | $\sum_n (n+1)e^{s_n} < \infty$ | $P(\varphi) = 0$ | yes | no | +ve recurrent | (iii) |
| | $\sum_n (n+1)e^{s_n} = \infty$ | $P(\varphi) = 0$ | no | yes | null recurrent | (iv) |
| $\sum_n e^{s_n} < 1$ | | $P(\varphi) = 0$ | no | yes | transient | (v) |

FIGURE 1. The first two columns summarise results in [H]: Equation (2.6) and Section 5. The final column applies Definition 2.5

.

We can make choices of $(a_n)_n$ so that the pressure function has the form:

$$p_\varphi(t) = \begin{cases} \text{strictly convex and real analytic} & \text{if } t \in [0,1), \\ 0 & \text{if } t > 1. \end{cases}$$

[2]The third entrance in the column of *Gibbs measures* was *no* in Hofbauer's [H] and it should have been *yes*.

The pressure is not analytic at $t = 1$. The real analyticity of the pressure for $t < 1$ follows, for instance, from [S3]. The general strategy to prove this type of result is to prove that the transfer operator is quasicompact. That is, to show that the essential spectral radius is strictly smaller than the spectral radius. This means that except for the spectrum inside a small disc the operator behaves like a compact operator (where the spectrum consists of isolated eigenvalues of finite multiplicity). Since the leading eigenvalue corresponds to the exponential of the pressure function, classic perturbation arguments allow for the proof of real analyticity of the pressure.

Moreover, we can choose $(a_k)_k$ so that:

- the map $t \mapsto P(t\varphi)$ is differentiable at $t = 1$ where $\varphi$ has only one equilibrium measure (the Dirac delta at zero). This is case (iv) in Figure 1;
- the map $t \mapsto P(t\varphi)$ has a first order phase transition at $t = 1$, i.e., is not differentiable at $t = 1$. Here and $\varphi$ has two equilibrium states, one is the Dirac delta at zero and the other can be seen as the projection of the Gibbs measure $\mu_\Phi$, the equilibrium state for $\Phi$. This is case (iii) in Figure 1.

Following [HK], for $\alpha > 0$ we could choose our sequence $(a_k)_k$ to be asymptotically like $\alpha \log\left(\frac{k+1}{k+2}\right)$ for all large $k$. The particular choice of $\alpha$ separates the two cases above.

The following lemma shows that by our definition of recurrence, the phase transition here corresponds to a switch from recurrence to transience. The lemma follows easily from [FL, Section 2], but we give a short sketch for completeness.

**Lemma 4.1.** *If $(a_k)_k$ are chosen so that $\sum_n e^{s_n} < 1$, then $(X, f, \varphi)$ is transient.*

*Proof.* Suppose that $\nu$ is a measure such that $L_\varphi^* \nu = e^{P(\varphi)}\nu = \nu$. Since $P(\varphi) = 0$ this implies that $\nu$ is a $(\varphi - P(\varphi))$-conformal measure. Note that $\nu$ must be a finite measure, so we will assume that it is a probability measure. We will show that such a $\nu$ does not exist.

We first observe that as in [FL, Theorem 2.2], $\nu$ must have no atomic part. However, as in [FL, Lemma 2.1], conformality implies that for $n \geqslant 1$, we have

$$1 = \nu([0,1)) = \nu(f^n(X_n)) = \int_{X_n} e^{-s_n} \, d\nu,$$

so $\nu(X_n) = e^{s_n}$ which implies

$$1 = \nu([0,1)) = \nu(\{0\}) + \sum_n e^{s_n}. \tag{5}$$

Therefore $\nu(\{0\}) > 0$, which is impossible, so there is no $(\varphi - P(\varphi))$-conformal measure. □

We also note that in the above proof, for the induced potential $\Phi$, we can show that $P(\Phi) < 0$, from which we can give an alternative proof that any $\varphi$-conformal measure must be supported on $\{f^{-n}(0)\}_{n \geqslant 0}$, and thus be atomic, using the techniques in Sections 5 and 6.

Now let us compare our definition of recurrence with (3). Let us suppose that $\sum_n e^{s_n} < 1$, so by the lemma above, our system is transient. The pressure here is clearly zero, so it would only make sense to put this value into the computation of recurrence in (3). If we choose our base set to be $[1/2, 1)$, then the definition of transience in (3) fits with ours. However, if we choose our base set to be, for example, $A := [0, 1/2)$, then

$$Z_n(\varphi, A) \geqslant e^{-nP(\varphi)} e^{S_n\varphi(0)} = 1, \tag{6}$$

which suggests that the system is actually recurrent. So blindly applying the computation of (3) here leads to recurrence being ill-defined. We note that the same kind of argument can be made against the application of Proposition 3.1.

4.2. **Manneville-Pomeau.** The following example was introduced by Manneville and Pomeau in [MP]. It is one of the simplest examples of a non-uniformly hyperbolic map. It is expanding and it has a parabolic fixed point at $x = 0$. For these systems and for the type of potential we choose below, there are the same issues with the definition of recurrence as in Section 4.1.

We give the form studied in [LSV]. For $\alpha > 0$, the map is defined by

$$f(x) = \begin{cases} x(1 + 2^\alpha x^\alpha) & \text{if } x \in [0, 1/2), \\ 2x - 1 & \text{if } x \in [1/2, 1). \end{cases} \tag{7}$$

The pressure function of the potential $-\log|f'|$ has the following form (see, for example, [Lo, S3]),

$$p(t) = \begin{cases} \text{strictly convex and real analytic} & \text{if } t \in [0, 1), \\ 0 & \text{if } t > 1, \end{cases}$$

where, for brevity we let

$$p(t) := P(-t \log|f'|).$$

(We use this notation throughout for this particular kind of potential.)

We first consider the potential $-\log|f'|$. Note that in all cases $p(1) = P(-\log|f'|) = 0$ and that Lebesgue is a $-\log|f'|$-conformal measure here. If an invariant probability measure is absolutely continuous w.r.t. Lebesgue, we call it an *acip*. It is well-known that if an acip exists in this setting, it is an equilibrium state for $-\log|f'|$. The value of $\alpha$ determines the class of differentiability of the map $f$ and determines the amount of time 'typical' orbits spend near the parabolic fixed point. Varying the value of $\alpha$ we obtain:

- $\alpha \in (0, 1)$: there exists and acip $\mu$. This corresponds to case (iii) in Figure 1 (positive recurrent).
- $\alpha \geqslant 1$: there is no acip. This corresponds to case (iv) in Figure 1 (null recurrent).

In the case $t > 1$, it is still true that $P(t) = 0$ for any $\alpha > 0$, but now, as in Lemma 4.1, there does not exist a $-t\log|f'|$-conformal measure. This corresponds to case (v) in Figure 1 and by our definition, the system is transient. Again, as in the argument around (6), our definition of recurrence/transience is more applicable here than (3).

**Remark 4.1.** *Note that the Dirac delta measure on* $0$ *is a conformal measure for* $-t\log|Df|$ *with* $t>1$ *if we remove all preimages of 0.*

4.3. **Chebyshev.** A simple example of a transitive map in the quadratic family which exhibits a phase transition is the Chebyshev polynomial $f(x):=4x(1-x)$ defined on $[0,1]$ (see for example [D]). For the set of *geometric potentials* $\{-t\log|f'|: t\in\mathbb{R}\}$, if $t>-1$ then the equilibrium state for $-t\log|f'|$ is the absolutely continuous (with respect to Lebesgue) invariant probability measure $\mu_1$, which has $\int\log|f'|\,d\mu_1=\log 2$. If $t<-1$ then the equilibrium state for $-t\log|f'|$ is the Dirac measure $\delta_0$ on the fixed point at 0, which has $\int\log|f'|\,d\delta_0=\log 4$. So, there exists a phase transition at $t_0=-1$ and

$$p(t)=\begin{cases}-t\log 4 & \text{if } t<-1,\\ (1-t)\log 2 & \text{if } t\geqslant -1.\end{cases}$$

For $t<-1$, if our base set $A$ includes the fixed point 0, then as in (6), we obtain $Z_n(\varphi,A)\geqslant 1$, which indicates recurrence. However, this situation should clearly not be thought of as recurrent. Indeed, the arguments in Section 5 can be adapted to show that this is transient by our definition.

4.4. **Multimodal maps.** Up until now, our examples have had 'bad' potentials, but the underlying dynamical system has nevertheless had some Markov structure. In this section we introduce a standard class of maps, many of which have no such structure. We study this class in more depth in Section 5.

Let $\mathcal{F}$ be the collection of $C^2$ multimodal interval maps $f:I\to I$, where $I=[0,1]$, satisfying:

a) the critical set $\mathcal{C}r=\mathcal{C}r(f)$ consists of finitely many critical points $c$ with critical order $1<\ell_c<\infty$, i.e., there exists a neighbourhood $U_c$ of $c$ and a $C^2$ diffeomorphism $g_c:U_c\to g_c(U_c)$ with $g_c(c)=0$ $f(x)=f(c)\pm|g_c(x)|^{\ell_c}$;
b) $f$ has negative Schwarzian derivative, i.e., $1/\sqrt{|Df|}$ is convex;
c) $f$ is topologically transitive on $I$;
d) $f^n(\mathcal{C}r)\cap f^m(\mathcal{C}r)=\emptyset$ for $m\neq n$.

For $f\in\mathcal{F}$ and $\mu\in\mathcal{M}_f$, let us define,

$$\lambda(\mu):=\int\log|f'|\,d\mu \qquad \text{and} \qquad \lambda_m:=\inf\{\lambda(\mu):\mu\in\mathcal{M}_f\}.$$

It was proved in [IT1] that there exists $t^+>0$ such that the pressure function of the (discontinuous) potential $\log|f'|$ satisfies,

$$p(t)=\begin{cases}\text{strictly convex and } C^1 & \text{if } t\in(-\infty,t^+),\\ At & \text{if } t>t^+.\end{cases}$$

In the case $\lambda_m=0$, $t^+\leqslant 1$ and $A=0$; while in the case $\lambda_m>0$, $t^+>1$ and $A<0$.

**Remark 4.2.** *The number of equilibrium measures at the phase transition can be large. Indeed, Cortez and Rivera-Letelier* [CRL] *proved that given $\mathcal{E}$ a non-empty, compact, metrisable and totally disconnected topological space then there exists a parameter $\gamma \in (0,4]$ such that set of invariant probability measures of $x \mapsto \gamma x(1-x)$, supported on the omega-limit set of the critical point is homeomorphic to $\mathcal{E}$. Examples of quadratic maps having multiple ergodic measures supported on the omega-limit set of the critical point were first constructed in* [Br].

Again (3) will not give us a reasonable way to check recurrence/transience for these maps due to the poor smoothness properties of the potential as well as the lack of Markov structure. Clearly, given what happened in the previous examples, we would expect that the phase transition at $t^+$ marked the switch from recurrence to transience, and indeed we show this in Section 5. However, as we show in Section 6 (see also [S3, Example 3]), it can happen that for systems $(X, f, t\varphi)$ increasing the parameter $t$ can take us from recurrence through transience and then out to recurrence again. So we should check the recurrence/transience of the systems in $\mathcal{F}$ outlined above. This is done Section 5.

4.5. **Brief summary of recurrence for interval maps.** All the examples of phase transitions presented in this section have the same type of behaviour. That is, the pressure function has one of the following two forms:

$$p_\varphi(t) = \begin{cases} \text{strictly convex and differentiable} & \text{if } t \in [0, t_0), \\ At & \text{if } t > t_0, \end{cases} \tag{8}$$

where $A \in \mathbb{R}$ is a constant. The regularity at the point $t = t_0$ varies depending on the examples. The other possibility is

$$p_\varphi(t) = \begin{cases} Bt + C & \text{if } t \in [0, t_0), \\ At & \text{if } t > t_0, \end{cases} \tag{9}$$

where $A, B, C \in \mathbb{R}$ are constants.

**Remark 4.3.** *It also possible for the pressure function to have the 'reverse form' to the one given in equation* (9)*: i.e., there are interval maps and potentials for which the pressure function has the form $p_\varphi(t) = At$ in an interval $(-\infty, t_0]$ and $p_\varphi(t) = Bt + C$ for $t > t_0$. The same 'reverse form' exists in the case that the pressure function is given as in equation* (8)*.*

Essentially what happens is that the dynamics can be divided into an hyperbolic part and a non-hyperbolic part (the latter having zero entropy, for example a parabolic fixed point or the post-critical set).

**Remark 4.4.** *As in Section* 3.3*, the situation can be completely different if the map is not assumed to be topologically mixing.*

A natural question that arises when considering the above examples is the following: Must the onset of transience always give pressure functions of the type in (8) or (9) (i.e., the onset of transience occurs 'at zero entropy' and once a potential is transient for some $t_0$ is either transient for all $t < t_0$ or $t > t_0$)? This is shown to be false in Section 6. In the case of shift spaces we point out [O] as well as the non-constructive Example 3 in [S3].

## 5. No conservative conformal measure

In this section we will show that the systems considered in Section 4 are transient past the phase transition. We focus on multimodal maps $f \in \mathcal{F}$ defined in Section 4.4. We will show that for a certain range of values of $t \in \mathbb{R}$ the potential $-t \log |f'|$ has no conservative conformal measure and hence is transient. The results described here also hold for the Manneville-Pomeau map, but since the proof is essentially the same, but simpler, we only discuss the former case. The first result deals with the recurrent case.

**Theorem 5.1.** *Suppose that $f \in \mathcal{F}$. If $t < t^+$ then there is a weakly expanding conservative $(-t \log |Df| - p(t))$-conformal measure.*

This is proved in the appendix of [T], where it is referred to as Proposition 7'. The idea of the proof is to obtain a conformal measure for an inducing scheme, as described below, and then spread this measure around the space in a canonical way.

**Proposition 5.1.** *Suppose that $f : I \to I$ belongs to $\mathcal{F}$ and $\lambda_m = 0$. Then for any $t > 1$, $(I, f, -t \log |Df|)$ is transient.*

This proposition covers the case when $t^+ = 1$. We expect this to also hold when $t^+ \neq 1$, but we do not prove this. As in Sections 4.1 and 4.2, the strategy used to study multimodal maps $f \in \mathcal{F}$, and indeed to prove Proposition 5.1, considering that they lack Markov structure and uniform expansivity, is to consider a generalisation of the first return map. These maps are expanding and are Markov (although over a countable alphabet). The idea is to study the 'inducing scheme' through the theory of Countable Markov Shifts and then to translate the results into the original system.

We say that $(X, \{X_i\}_i, F, \tau) = (X, F, \tau)$ is an *inducing scheme* for $(I, f)$ if

- $X$ is an interval containing a finite or countable collection of disjoint intervals $X_i$ such that $F$ maps each $X_i$ diffeomorphically onto $X$, with bounded distortion (i.e. there exists $K > 0$ so that for all $i$ and $x, y \in X_i$, $1/K \leqslant DF(x)/DF(y) \leqslant K$);
- $\tau|_{X_i} = \tau_i$ for some $\tau_i \in \mathbb{N}$ and $F|_{X_i} = f^{\tau_i}$. If $x \notin \cup_i X_i$ then $\tau(x) = \infty$.

The function $\tau : \cup_i X_i \to \mathbb{N}$ is called the *inducing time*. It may happen that $\tau(x)$ is the first return time of $x$ to $X$, but that is certainly not the general case. We denote the set of points $x \in I$ for which there exists $k \in \mathbb{N}$ such that $\tau(F^n(f^k(x))) < \infty$ for all $n \in \mathbb{N}$ by $(X, F, \tau)^\infty$.

The space of $F-$invariant measures is related to the space of $f$-invariant measures. Indeed, given an $f$-invariant measure $\mu$, if there is an $F$-invariant measure $\mu_F$ such that for a subset $A \subset I$,

$$\mu(A) = \frac{1}{\int \tau \, d\mu_F} \sum_{k=1}^{\infty} \sum_{i=0}^{k-1} \mu_F \left( f^{-k}(A) \cap X_i \right) \tag{10}$$

where $\frac{1}{\int \tau \, d\mu_F} < \infty$, we call $\mu_F$ the *lift* of $\mu$ and say that $\mu$ is a *liftable* measure. Conversely, given a measure $\mu_F$ that is $F$-invariant we say that $\mu_F$ *projects* to $\mu$ if (10) holds. We say that a $f-$invariant probability measure $\mu$ is *compatible* with the inducing scheme $(X, F, \tau)$ if

- We have that $\mu(X) > 0$ and $\mu(X \setminus (X, F)^{\infty}) = 0$, and
- there exists a $F-$invariant measure $\mu_F$ which projects to $\mu$

**Remark 5.1.** *Let $\mu$ be a liftable measure and be $\nu$ be its lift. A classical result by Abramov* [A] *(see also* [PS, Z]*) allow us to relate the entropy of both measures. Further results obtained in* [PS, Z] *allow us to do the same with the integral of a given potential $\varphi : I \to \mathbb{R}$. Indeed, for the induced potential $\Phi$ we have that*

$$h(\mu) = \frac{h(\nu)}{\int \tau \, d\nu} \text{ and } \int \varphi \, d\mu = \frac{\int \Phi \, d\nu}{\int \tau \, d\nu}.$$

*Also a $\varphi$-conformal measure $m_\varphi$ for $(I, f)$ is also a $\Phi$-conformal measure for $(X, F)$ if $m_\varphi(\cup X_i) = m_\varphi(X)$.*

For $f \in \mathcal{F}$ we choose the domains $X$ to be $n$-cylinders coming from the so-called *branch partition*: the set $\mathcal{P}_1^f$ consisting of maximal intervals on which $f$ is monotone. So if two domains $C_1^i, C_1^j \in \mathcal{P}_1^f$ intersect, they do so only at elements of $\mathcal{C}r$. The set of corresponding $n$-cylinders is denoted $\mathcal{P}_n^f := \vee_{k=1}^n f^{-k}\mathcal{P}_1$. We let $\mathcal{P}_0^f := \{I\}$. For an inducing scheme $(X, F, \tau)$ we use the same notation for the corresponding $n$-cylinders $\mathcal{P}_n^F$.

The following result, proved in [T] (see also [BT2, IT1]) proves that useful inducing schemes exist for maps $f \in \mathcal{F}$.

**Theorem 5.2.** *Let $f \in \mathcal{F}$. There exist a countable collection $\{(X^n, F_n, \tau_n)\}_n$ of inducing schemes with $\partial X^n \notin (X^n, F_n, \tau_n)^{\infty}$ such that any ergodic invariant probability measure $\mu$ with $\lambda(\mu) > 0$ is compatible with one of the inducing schemes $(X^n, F_n, \tau_n)$. Moreover, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $LS_\varepsilon \subset \cup_{n=1}^N (X^n, F_n, \tau_n)^{\infty}$.*

We are now ready to apply this theory to the question of transience, building up to proving Proposition 5.1.

**Lemma 5.1.** *Suppose that $f \in \mathcal{F}$. If, for $t > t^+$, there is a conservative weakly expanding $(-t \log |Df| - s)$-conformal measure $m_{t,s}$ for some $s \in \mathbb{R}$, then $s \leqslant P(-t \log |Df|)$. Moreover, there is an inducing scheme $(X, F, \tau)$ such that*

$$P(-t \log |DF| - \tau s) = 0$$

*and*

$$m_{t,s} \left( \left\{ x \in X : \tau^k(x) \text{ is defined for all } k \geqslant 0 \right\} \right) = m_{t,s}(X).$$

*Proof.* We prove the second part of the lemma first.

Suppose that $m_{t,s}$ is a weakly expanding $(-t\log|Df|-s)$-conformal measure. We introduce an inducing scheme $(X,F)$. Since $m_{t,s}$ is weakly expanding, by Theorem 5.2 there exists an inducing scheme $(X,F,\tau)$ such that

$$m_{t,s}\left(\left\{x\in X:\tau^k(x)\text{ is defined for all }k\geqslant 0\right\}\right)=m_{t,s}(X)>0. \tag{11}$$

This can be seen as follows. Theorem 5.2 implies that there exists $(X,F)$ with $m_{t,s}((X,F)^\infty)>0$. If $m_{t,s}(X\setminus(X,F)^\infty)>0$ then there must exist $k\in\mathbb{N}$ such that the set

$$A_k:=\left\{x\in X:\exists\ i(x)\in\mathbb{N}\text{ such that }\tau^{i(x)}(x)=k\text{ but }\tau(F^{i(x)})=\infty\right\}$$

has $m_{t,s}(A_k)>0$. But a standard argument shows that $A_k$ is a wandering set: if not then there exist $n>j\in\mathbb{N}_0$ such that there is a point $x\in f^{-n}(A_k)\cap f^{-j}(A_k)$: so $z:=f^j(x)\in A_k$ and $f^{n-j}(z)\in A_k$, which is impossible.

By the distortion control for the inducing scheme, for any $n$-cylinder $\mathbf{C}_{n,i}\in\mathcal{P}_n^F$ and since $m_{t,s}(X)=\int_{\mathbf{C}_{n,i}}|DF^n|^te^{s\tau^n}\,dm_t$, there exists $K\geqslant 1$ such that

$$|\mathbf{C}_{n,i}|^te^{s\tau^n}=K^{\pm t}|X|^tm_{t,s}(\mathbf{C}_{n,i}). \tag{12}$$

Since the inducing scheme is the full shift, and because of this distortion property, the pressure of $-t\log|DF|-s\tau$ can be computed as

$$\lim_{n\to\infty}\frac{\log\left(\sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F}|\mathbf{C}_{n,i}|^te^{s\tau^n}\right)}{n}.$$

However, using first (12) and then (11), we have

$$\sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F}|\mathbf{C}_{n,i}|^te^{s\tau^n}=K^{\pm t}|X|^t\sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F}m_{t,s}(\mathbf{C}_{n,i})=K^{\pm t}|X|^tm_{t,s}(X)$$

for all $n\geqslant 1$. This implies that $P(-t\log|DF|-s\tau)=0$, proving the second part of the lemma.

We prove the first part of the lemma by applying the Variational Principle to the inducing scheme. Since $P(-t\log|DF|-s\tau)=0$, by [S1, Theorem 2], there exists a sequence $(\mu_{F,n})_n$ each supported on a finite number of cylinders in $\mathcal{P}_1^F$ and with

$$\lim_{n\to\infty}\left(h(\mu_{F,n})+\int -t\log|DF|-s\int\tau\,d\mu_{F,n}\right)=0.$$

Therefore, by the Abramov Theorem (see Remark 5.1), for the projected measures $\mu_n$ we have

$$h(\mu_n)-\int t\log|Df|\,d\mu_n\to s.$$

Hence the definition of pressure implies that $s\leqslant p(t)$. □

*Proof of Proposition* 5.1. Suppose that there exists a weakly expanding conservative $-t\log|Df|$-conformal measure $m_t$. Let $(X,F)$ be the inducing scheme in

Lemma 5.1, with distortion $K \geqslant 1$. Then $P(\Psi_t) = 0$ and

$$\begin{aligned} m_t(X) &= \sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} m_t(\mathbf{C}_{n,i}) = K^{\pm t}|X|^t \sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} |\mathbf{C}_{n,i}|^t \\ &= K^{\pm t}|X|^t \sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} |\mathbf{C}_{n,i}||\mathbf{C}_{n,i}|^{t-1} \\ &\leqslant K^t|X|^t \left(\sup_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} |\mathbf{C}_{n,i}|\right)^{t-1} \sum_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} |\mathbf{C}_{n,i}| \\ &\leqslant K^t|X|^{t+1} \left(\sup_{\mathbf{C}_{n,i}\in\mathcal{P}_n^F} |\mathbf{C}_{n,i}|\right)^{t-1}. \end{aligned}$$

Since $t > 1$ by choosing $n$ large, we can make this arbitrarily small, so we are led to a contradiction. □

## 6. Possible transient behaviours

In this section we address some of the questions raised about the possible behaviours of transient systems in Section 2. In particular, we present an example which gives us a range of possible behaviours for a pressure function which has one or two phase transitions. This example is very similar to that presented by Olivier in [O, Section 4] in which he extended the ideas of Hofbauer [H] (also considered in detail in [Lo, FL]) to produce a system with hyperbolic dynamics, but with a potential $\varphi$ which was sufficiently irregular to produce a phase transition: the support of the relevant equilibrium states $t\varphi$ jumping from the whole space to an invariant subset as $t$ moved through the phase transition. We follow the same kind of argument, with slightly simpler potentials. In our case, we are able to obtain very precise information on the pressure function and on the measures at the phase transition. Moreover, we can arrange our system so that the support of the relevant equilibrium state for $t\varphi$ jumps from the whole space, to an invariant subset, and then back out to the whole space as $t$ increases from $-\infty$ to $\infty$. Between the phase transitions we have transience.We point out that Sarig proved the existence of such phenomena in [S3], but here we are able to give an explicit, and fairly elementary, construction.

**Definition 6.1.** *For a dynamical system $(X, f)$ with a potential $\varphi$, let us consider conditions i) $\lim_{t\to-\infty} p_\varphi(t) = \infty$; ii) there exist $t_1 < t_2$ such that $p_\varphi(t)$ is constant on $[t_1, t_2]$; iii) $\lim_{t\to\infty} p_\varphi(t) = \infty$. We say that $p_\varphi$ is* DF *(for down-flat) if i) and ii) hold; that $p_\varphi$ is* DU *(for down-up) if i) and iii) hold; that $p_\varphi$ is* DFU *(for down-flat-up) if i), ii) and iii) hold.*

In this section we describe a situation with pressure which is DFU. The system is the full-shift on three symbols $(\Sigma_3, \sigma)$. (Note that we could instead consider $\{I_i\}_{i=1}^3$, three pairwise disjoint intervals contained in $[0, 1]$, and the map $f : \bigcup_{i=1}^3 I_i \subset [0, 1] \to [0, 1]$, where $f(I_i) = [0, 1]$ which is topologically (semi-)conjugated to $(\Sigma_3, \sigma)$.) The construction we will use can be thought of as a generalisation of the renewal shift (see Section 3.2). Let $(\Sigma_3, \sigma)$ be the full shift on three symbols $\{1, 2, 3\}$. A point $x \in \Sigma_3$ can be written as $x = (x_0x_1x_2 \dots)$, where

$x_i \in \{1,2,3\}$. Our *bad set* (the we will denote by $B$) will be the full shift on two symbols $\{1,3\}$ and the *renewal vertex* will be $\{2\}$.

For $N \geqslant 1$ and $(x_0, x_1, \dots, x_{N-1}) \in \Sigma_3$, let $[x_0 x_1 \dots x_{N-1}]$ denote the cylinder $C_{x_0 x_1 \dots x_{N-1}}$. We set $M_0$ to be the cylinder [2] and define the *first hitting time* to [2] as the function $\tau : \Sigma_3 \to \mathbb{N}$ defined by $\tau(x) = \inf\{n \in \mathbb{N} : \sigma^n x \in [2]\}$.

We set

$$M_n := \{x \in \Sigma_3 : \tau(x) = n\}.$$

The class of potentials is given as follows.

**Definition 6.2.** *A function $\varphi : \Sigma_3 \to \mathbb{R}$ is called a* grid function *if it is of the form*

$$\varphi(x) = \sum_{n=0}^{\infty} a_n \cdot \mathbb{1}_{M_n}(x),$$

*where $\mathbb{1}_{M_n}(x)$ is the characteristic function of the set $M_n$ and $(a_n)_{n\in\mathbb{N}_0}$ is a sequence of real numbers such that $\lim_{n\to\infty} a_n = 0$. Note that $\varphi|_B = 0$.*

Grid functions were introduced in a more general form by Markley and Paul [MPa]: they allowed the set $B$ to be any subshift and $X_n$ to be any partition elements converging to $B$ in the Hausdorff metric. They were presented as a generalisation of those functions by Hofbauer [H] which we described in Section 4.1. They can be thought of as weighted distance functions to a *bad* set $B$. Recently, this type of potential was used to disprove an ergodic optimisation conjecture [ChH].

**Remark 6.1.** *The measure of maximal entropy $\mu_B$ on $B$ has $h(\mu_B) = \log 2$. Since $\varphi|_B \equiv 0$, this implies that for any $t \in \mathbb{R}$, $p_\varphi(t) \geqslant h(\mu_B) + \int t\varphi \, d\mu_B = h(\mu_B) = \log 2$.*

We are now ready to state our result concerning the thermodynamic formalism for our grid functions.

**Theorem 6.1.** *Let $(\Sigma_3, \sigma)$ be the full-shift on three symbols and let $\varphi : \Sigma_3 \mapsto \mathbb{R}$ be a grid function defined by a sequence $(a_n)_n$. Then*

1. *there exist $(a_n)_n$ so that $D^- p_\varphi(1) < 0$, but $p_\varphi(t) = \log 2$ for all $t \geqslant 1$;*
2. *there exist $(a_n)_n$ and $t_1 > 1$ so that $D^- p_\varphi(1) < 0$, $p_\varphi(t) = \log 2$ for all $t \in [1, t_1]$ and $Dp_\varphi(t) > 0$ for all $t > t_1$;*
3. *there exist $(a_n)_n$ so that $Dp_\varphi(t) < 0$ for $t < 1$, but $p_\varphi$ is $C^1$ at $t = 1$ and $p_\varphi(t) = \log 2$ for all $t \geqslant 1$;*
4. *there exist $(a_n)_n$ and $t_1 > 1$ so that $Dp_\varphi(t) < 0$ for $t < 1$, but $p_\varphi$ is $C^1$ at $t = 1$, and $p_\varphi(t) = \log 2$ for all $t \in [1, t_1]$ and $Dp_\varphi(t) > 0$ for all $t > t_1$;*

We comment further on the systems $(\Sigma_3, \sigma, t\varphi)$ with reference to Table 1 (note that the only aspects which don't follow more or less immediately from the construction of our sequences $(a_n)_n$ are the statements about recurrence and differentiability at $t = 1, t_1$, which follow from Lemma 6.3 and Proposition 6.1):

- In case (1) of the theorem, the system is positive recurrent for $t \leqslant 1$ and transient for $t > 1$. The pressure function $p_\varphi$ is DF. See the left hand side of Figure 2.

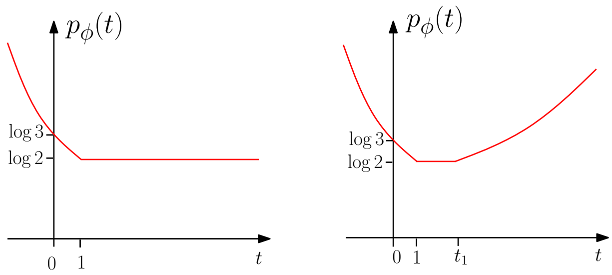


FIGURE 2. Sketch of cases (1) and (2) of Theorem 6.1.

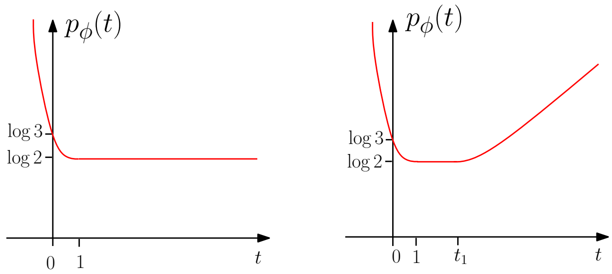


FIGURE 3. Sketch of cases (3) and (4) of Theorem 6.1.

- In case (2) of the theorem, the system is positive recurrent for $t \in (-\infty, 1] \cup [t_1, \infty)$ and transient for $t \in (1, t_1)$. The pressure function is DFU. See the right hand side of Figure 2.
- In case (3) of the theorem, the system is positive recurrent for $t < 1$, null recurrent for $t = 1$ and transient for $t > 1$. The pressure function is DF. See the left hand side of Figure 3.
- In case (4) of the theorem, the system is positive recurrent for $t \in (-\infty, 1) \cup (t_1, \infty)$, null recurrent for $t = 1, t_1$ and transient for $t \in (1, t_1)$. The pressure is DFU. See the right hand side of Figure 3.
- In the cases above where $t\varphi$ is transient, as in Lemma 4.1, there is a dissipative $t\varphi$-conformal measure.

Note that in the case of Hofbauer's example, described in 4.1, the modes of recurrence of the potential and behaviour of the pressure function are determined by the form of the sums of the sequence $(a_n)_n$ (see Table 1). This is also the case in our example.

6.1. **The inducing scheme.** For $n \in \mathbb{N}$, let $\{X_n^i\}_i$ denote the connected components in [2] with $\tau$ equal to $n$. This is a set of $2^{n-1}$ cylinders. For example,

$$\{X_1^i\}_i = \{[22]\}$$

$$\{X_2^i\}_i = \{[212], [232]\}$$

$$\{X_3^i\}_i = \{[2112], [2132], [2312], [2332]\}.$$

The *first return map*, denoted by $F$, is defined by
$$F(x) = \sigma^n(x) \text{ if } x \in X_n^i.$$
Note that the bad set $B$, on which $F$ is not defined, can be thought of as a coding for the middle third Cantor set.

The induced potential for this first return map is given by $\Phi(x) = S_{\tau(x)}\varphi(x)$. For $n \geqslant 1$ we let
$$s_n := a_0 + \cdots + a_{n-1}.$$
Then by definition of $\varphi$, for any $x \in X_n^i$ we have $\Phi(x) = s_n$.

The definitions of liftability of measures and aspects of inducing schemes for the case considered here are directly analogous to the setting considered in Section 5, so we do not give them here.

**Remark 6.2.** *Note that the potential $\Phi$ is a locally constant over the countable Markov partition $\bigcup_{n,i} X_n^i$. Therefore with the inducing procedure we have gained regularity on our potential.*

Given a grid function (our potential) $\varphi$ defined on $\Sigma_3$, to discuss equilibrium states for the induced system, as in Section 5, it is convenient to shift the original potential to ensure that its induced version will have pressure zero. Therefore, since we will be interested in the family of potentials $t\varphi$ with $t \in \mathbb{R}$, we set
$$\psi_t := t\varphi - p_\varphi(t) \text{ and } \Psi_t := t\Phi - \tau p_\varphi(t).$$
Note that $x \in X_n^i$ implies $\Psi_t(x) = ts_n - np_\varphi(t)$. We denote this value $\Psi_t|_{X_n^i}$ by $\Psi_{t,n}$.

**Remark 6.3.** *As in* [Ba, p25] *for example, since $\Psi_t$ is locally constant,*
$$e^{P(\Psi_t)} = \sum_n 2^{n-1} e^{\Psi_{t,n}}.$$
*Moreover, if $P(\Psi_t) = 0$ then there exists a Gibbs measure $\mu_{\Psi_t}$ which has $\mu_{\Psi_t}(X_n^i) = e^{\Psi_{t,n}}$ (see* [PU, Lemma 4.4.2]*) and so is both a $\Psi_t$-conformal measure and an invariant measure. Thus, if $-\int \Psi_t \, d\mu_{\Psi_t} < \infty$ then $\mu_{\Psi_t}$ is also an equilibrium state. Since $\psi_t$ is bounded, the Abramov formula implies that this occurs when $\int \tau \, d\mu_{\Psi_t} = \sum_n n 2^{n-1} e^{\Psi_{t,n}} < \infty$.*

We also have the following result, similarly to Lemma 5.1, and which also follows in this case from [S3, Lemma 3].

**Lemma 6.1.** *There exists a conservative $\psi_t$-conformal measure $m_t$ if and only if $P(\Psi_t) = 0$.*

*Proof.* First suppose that there is a $\psi_t$-conformal measure $m_t$. Then since $\sigma[2] = \Sigma_3$, and $m_t(\Sigma_3) = 1$, we have $m_t([2]) = e^{a_0} > 0$. As in Remark 6.3, $m_t(X_n^i) = m_t([2])e^{\Psi_{t,n}}$. Therefore,
$$e^{P(\Psi_t)} = \sum_n 2^{n-1} e^{\Psi_{t,n}} = \frac{1}{m_t([2])} \sum_{n,i} m_t(X_n^i) = 1,$$
so $P(\Psi_t) = 0$.

Now suppose that $P(\Psi_t) = 0$. Then [S2, Theorem 1] implies that there is a $\Psi_t$-conformal measure $\tilde{m}_t$. In particular, if $A \subset X_n^i$ then

$$\tilde{m}_t(A) = \tilde{m}_t(f^n(A))e^{\Psi_{t,n}} = \tilde{m}_t(f^n(A))e^{ts_n - np(t)}.$$

We extend $\tilde{m}_t$ to the rest of $\Sigma_2$ as follows. If $A \subset M_n \setminus [2]$ then define

$$\tilde{m}_t(A) = e^{t(a_n + a_{n-1} + \cdots + a_1) - np(t)}\tilde{m}(f^n(A)).$$

We check the $\psi_t$-conformality of this measure. Notice that by this definition if $k \leqslant n-1$, then

$$\tilde{m}_t(f^k(A)) = e^{t(a_{n-k} + a_{n-k-1} + \cdots + a_1) - (n-k)p(t)} = \int_A e^{-S_k\psi_t}\, d\tilde{m}_t.$$

Moreover, if $A \subset X_n^i$ then this definition gives

$$\begin{aligned}
\tilde{m}_t(f(A)) &= e^{t(a_{n-1} + a_{n-2} + \cdots + a_1) - (n-1)p(t)}\tilde{m}_t(f^n(A)) \\
&= \tilde{m}_t(f^n(A))e^{ts_n - np(t)}e^{-ta_0 + p(t)} = \tilde{m}_t(A)e^{-ta_0 + p(t)} \\
&= \int_A e^{-\psi_t}\, d\tilde{m}_t,
\end{aligned}$$

which confirms the conformality of $\tilde{m}_t$. Note that $\tilde{m}_t(\Sigma_3) = e^{ta_0}$, so we can normalise $\tilde{m}_t$ if necessary. □

In this setting, the properties of the pressure function will depend directly on the choice of the sequence $(a_n)_n$. Our first requirement is that the system to be recurrent at $t = 1$ and to have $p_\varphi(t) = P(\varphi) = \log 2$. Thus Remark 6.3 and Lemma 6.1 imply that we should choose $(a_n)_n$ so that

$$1 = \sum_i e^{\Psi_{1,i}} = \sum_{n \geqslant 1} 2^{n-1}e^{s_n - n\log 2} = \frac{1}{2}\sum_{n \geqslant 1} e^{s_n}. \tag{13}$$

6.2. **Down-flat pressure occurs.** For every $n \in \mathbb{N}_0$ the numbers $a_n$, are chosen so that $a_n < 0$ and (13) holds. Since $\varphi \leqslant 0$; the pressure function $p_\varphi$ is non-increasing in $t$; $p_\varphi(1) = \log 2$; and $p_\varphi(t) \geqslant \log 2$ (Remark 6.1); this means that $p_\varphi(t) = \log 2$ for all $t \geqslant 1$, so the DF case occurs.

This gives the grounding for parts (1) and (3) of Theorem 6.1. We prove the results on the derivative of the pressure function below.

6.3. **Down-flat-up pressure occurs.** For the DFU case, we start with $(a_n)_n$ as in the DF case above with the exception that $a_0$ is set to be 0. Since in this case $s_1 = 0$, (13) can be rewritten as

$$1 = \frac{1}{2}\sum_{n \geqslant 1} e^{s_n} = \frac{1}{2}\left(1 + \sum_{n \geqslant 2} e^{s_n}\right). \tag{14}$$

Next we replace $a_0$ by $\tilde{a}_0 := \delta \in (0, \log 2)$, and $a_1$ by $\tilde{a}_1 := a_1 + \delta'$, where $\delta' < 0$ is such that (14) still holds when $(s_n)_n$ is replaced by $(\tilde{s}_n)_n$, the rest of the $a_n$ being kept fixed. So (14) implies that

$$\frac{1}{2}e^{\delta} + \frac{1}{2}e^{\delta+\delta'} = 1,$$

so $P(\Psi_1) = 0$. Using Taylor series, we have $2\delta + \delta' < 0$. We now replace $\varphi$, $\Phi$ and $\Psi_t$ by the adjusted potentials $\tilde{\varphi}$, $\tilde{\Phi}$, $\tilde{\Psi}_t$.

**Lemma 6.2.** *We can choose $(\tilde{a}_n)_n$ as above so that there exists $t_1 > 1$ such that $P(\tilde{\Psi}_t) < 0$ for all $t \in (1, t_1)$.*

Since $p_{\tilde{\varphi}}(t) \geqslant \log 2$, the lemma implies that $p_{\tilde{\varphi}}(t) = \log 2$ for $t \in [1, t_1]$, so the DF property of the pressure function persists under our perturbation of $\varphi$ to $\tilde{\varphi}$. Moreover, Lemma 6.1 implies that $(\Sigma, \sigma, t\tilde{\varphi})$ is transient for $t \in [1, t_1]$.

The 'up' part of the DFU property, must hold for $p_{\tilde{\varphi}}$ since $\tilde{a}_0 > 0$: indeed the graph of $p_{\tilde{\varphi}}$ must be asymptotic to $t \mapsto \tilde{a}_0 t$, and the equilibrium measures for $t\tilde{\varphi}$ denoted by $\mu_t$ must tend to the Dirac measure on the fixed point in [2].

*Proof of Lemma* 6.2. As above, since $\tilde{\Psi}_t$ is locally constant, $e^{P(\tilde{\Psi}_t)}$ can be computed as

$$e^{P(\tilde{\Psi}_t)} = \sum_{i\geqslant 1} e^{t\tilde{\Phi}_i - \tau_i p_{\tilde{\varphi}}(t)} = \sum_{n\geqslant 1} 2^{n-1} e^{t\tilde{s}_n - np_{\tilde{\varphi}}(t)} \leqslant \frac{1}{2}\sum_{n\geqslant 1} e^{t\tilde{s}_n},$$

as $p_{\tilde{\varphi}}(t) \geqslant \log 2$. Since, moreover, $s_n < 0$ for $n \geqslant 2$, if $t > 1$ is close to 1 and $\delta > 0$ is close to zero, then

$$e^{P(\tilde{\Psi}_t)} \leqslant \frac{1}{2}\sum_{n\geqslant 1} e^{t\tilde{s}_n} = \frac{1}{2}\left(e^{t\delta} + e^{t(\delta+\delta')}\sum_{n\geqslant 2} e^{ts_n}\right) < \frac{1}{2}\left(e^{t\delta} + e^{t(\delta+\delta')}\right) < 1,$$

where the final inequality follows from a Taylor series expansion and the fact that $2\delta + \delta' < 0$. This implies that $P(\tilde{\Psi}_t) < 0$ for $t > 1$ close to 1. We let $t_1 > t'' > 1$ be minimal such that $t > t_1$ implies $p_{\tilde{\varphi}}(t) > \log 2$. □

This proves that we can choose our potentials so that the pressure has the essential form described in parts (2) and (4) of Theorem 6.1. It remains to prove the claims of the derivatives of the pressure and the modes of recurrence. For brevity, from here on we will drop the tildes from our notation when discussing the potentials above.

6.4. **Tails and smoothness.** So far we have not made any assumptions on the precise form of $a_n$ for large $n$. In this section we will make our assumptions precise in order to distinguish cases (1) from case (3) in Theorem 6.1, as well as case (2) from case (4). That is to say, we will address the question of the smoothness of $p_\varphi$ at 1 and $t_1$ by defining different forms that $a_n$, and hence $s_n$, can take as $n \to \infty$. In fact, it is only the form of $a_n$ for large $n$ which separates the cases we consider. As in [H, Section 4], see also [Lo, Section 2] and [BT1, Section 6], let us assume that for some $\gamma > 1$ and for all large $n$, we have

$$a_n = \gamma \log\left(\frac{n}{n+1}\right).$$

We will see that we have a first order phase transition in the pressure function $p_\varphi$ whenever $\gamma > 2$, but not when $\gamma \in (1, 2]$.

Clearly, there is some $\kappa \in \mathbb{R}$ so that $s_n \sim \kappa - \gamma \log n$. So applying the computation in (13),

$$\sum_i e^{\Psi_{1,i}} = \frac{1}{2} \sum_n e^{s_n} = (1 + O(1)) \sum_n \frac{1}{n^\gamma}. \tag{15}$$

Since we assumed that $\gamma > 1$, we can ensure that this is finite, and indeed we can choose $(a_n)_n$ in such a way that $\sum_i e^{\Psi_{1,i}} = 1$ as in (13).

**Lemma 6.3.** *Suppose that $\varphi$ is a grid function as above.*

I. *If $P(\Psi_t) = 0$ and*
   (a) *$\sum_n n2^{n-1} e^{\Psi_{t,n}} < \infty$ then there exists an equilibrium state $\mu_t \ll m_t$ and the system is positive recurrent.*
   (b) *$\sum_n n2^{n-1} e^{\Psi_{t,n}} = \infty$ then there is no equilibrium state $\mu_t \ll m_t$ and the system is null recurrent.*

II. *If $P(\Psi_t) < 0$ then the system is transient.*

*Proof.* Case II follows as in Lemma 5.1 for example. In case I, as in Remark 6.3, there exists an $F$-invariant $\Psi_t$-conformal measure $\mu_{\Psi_t}$. Moveover, by Lemma 6.1, this produces a conservative $\psi_t$-conformal measure $m_t$.

Case I(a) is standard since $P(\Psi_t) = 0$ and $\sum_n n2^{n-1} e^{\Psi_{t,n}} < \infty$ imply that the measure $\mu_{\Psi_t}$ projects to an invariant probability measure $\mu_t \ll m_t$. The fact that $\mu_t$ is an equilibrium state follows from the Abramov formula.

For case I(b), suppose that $P(\Psi_t) = 0$ and that there exists an equilibrium state $\mu_t \ll m_t$ for $\psi_t$. This implies that $\mu_t([2]) > 0$, so by Kac's Lemma, the measure $\mu_{t,F} = \mu_t / \mu_t([2])$ is the lift of $\mu_t$, in particular $\int \tau \, d\mu_{t,F} = \frac{1}{\mu_t([2])} < \infty$. By the Abramov formula, $h(\mu_{t,F}) + \int \Psi_t \, d\mu_{t,F} = 0 = P(\Psi_t)$, so $\mu_{t,F}$ is the unique equilibrium state for $\Psi_t$. Remark 6.3 implies that $\mu_{t,F} = \mu_{\Psi_t}$. Thus the equation $\int \tau \, d\mu_{t,F} < \infty$ becomes $\sum_n n2^{n-1} e^{\Psi_{t,n}} < \infty$, so the existence of such a $\mu_t$ only occurs in case I(a). □

Since $p(t) > \log 2$ implies that $2^{n-1} e^{\Psi_{t,n}}$ is exponentially small in $n$, Lemma 6.3 implies that we are always in case I(a). The other cases depend on the value of $\gamma$: as in (15), for $\gamma > 2$ we are in case I(a); for $\gamma = 2$, we are in case I(b); for $\gamma \in (1, 2)$, we are in case II.

We now show that the graph of the pressure in the case that the pressure is DFU is either $C^1$ everywhere or only non-$C^1$ at both $t = 1$ and $t = t_1$. The issue of smoothness of $p_\varphi$ in the $DF$ case follows as in the DFU case, so Theorem 6.1 then follows from Lemma 6.2 and the following proposition.

**Proposition 6.1.** *For potential $\varphi$ chosen as above, there exists $t_1 > 1$ such that $p_\varphi(t) = \log 2$ for all $t \in [1, t_1]$. Moreover, if $\gamma \in (1, 2]$ then $p_\varphi$ is everywhere $C^1$, while if $\gamma > 2$ then $p_\varphi$ fails to be differentiable at both $t = 1$ and $t = t_1$.*

The first part of the proposition follows from Lemma 6.2, while the second follows directly from the following two lemmas. We will use the fact that if $p_\varphi$ is $C^1$ at $t$ then $Dp_\varphi(t) = \int \varphi \, d\mu_t$ (see [PU, Chapter 4]).

**Lemma 6.4.** *If $\gamma \in (1,2]$ then $Dp_\varphi(1) = 0$.*

*Proof.* Since by Lemma 6.2, for $t \in [1, t_1]$, $p_\varphi$ is constant $\log 2$, we have $Dp_\varphi^+(1) = 0$, so to prove $Dp_\varphi(1) = 0$ we must show $Dp_\varphi^-(1) = 0$. We use the fact that if $p_\varphi$ is $C^1$ at $t$, then $Dp_\varphi(t) = \int \varphi \, d\mu_t$.

Suppose that $t < 1$. Then by the Abramov formula (see Remark 5.1),

$$\int \varphi \, d\mu_t = \frac{\int \Phi \, d\mu_{\Psi_t}}{\int \tau \, d\mu_{\Psi_t}} = \frac{\sum_n s_n e^{ts_n - n(p_\varphi(t) - \log 2)}}{2\sum_n n e^{ts_n - n(p_\varphi(t) - \log 2)}}. \tag{16}$$

As above, for large $n$, $s_n \sim \kappa - \gamma \log n$, which is eventually much smaller, in absolute value, than $n$. Since also, $\sum_n n e^{s_n - n(p_\varphi(t) - \log 2)} \to \infty$ as $t \to 1$, we can make $\int \varphi \, d\mu_t$ arbitrarily small by taking $t < 1$ close enough to 1. Since when $p_\varphi$ is $C^1$ at $t$ then $Dp_\varphi(t) = \int \varphi \, d\mu_t$, this completes the proof. □

**Lemma 6.5.** *Suppose that $\varphi$ is a grid function as above and the pressure $p_\varphi$ is DFU. Then $p_\varphi$ is $C^1$ at $t = 1$ if and only if $p_\varphi$ is $C^1$ at $t = t_1$.*

*Proof.* The argument in the proof of Lemma 6.4, in particular (16), implies that if $t \in \mathbb{R}$ has $\int \tau \, d\mu_{\Psi_t} = \infty$, then we can make $Dp_\varphi(t')$ arbitrarily close to 0 by taking $t'$ close enough to $t$. Similarly if this integral is finite at $t$ then the derivative $Dp_\varphi(t)$ is non-zero. So to prove the lemma, we need to show that the finiteness or otherwise of $\int \tau \, d\mu_{\Psi_t}$ is the same at both $t = 1$ and $t = t_1$.

As in the proof of Lemma 6.2, $s_n < 0$ for $n \geqslant 2$. So since $p_\varphi(1) = p_\varphi(t_1)$ and $t_1 > 1$,

$$\int \tau \, d\mu_{\Psi_1} > \sum_{i \geqslant 2} \tau_i e^{\Phi_i - \tau_i p_\varphi(1)} > \sum_{i \geqslant 2} \tau_i e^{t_1 \Phi_i - \tau_i p_\varphi(t_1)} = \int_{X \setminus X_1} \tau \, d\mu_{\Psi_{t_1}}.$$

Therefore if $\int \tau \, d\mu_{\Psi_1} < \infty$ then $\int \tau \, d\mu_{\Psi_{t_1}} < \infty$. Similarly, if $\int \tau \, d\mu_{\Psi_{t_1}} = \infty$ then $\int \tau \, d\mu_{\Psi_1} = \infty$. Hence either $Dp_\varphi(1)$ and $Dp_\varphi(t_1)$ are both 0 or are both non-zero. □

**Remark 6.4.** *In the case $\gamma \in (1,2]$, the measure $\mu_{\Psi_1}$ is not regarded as an equilibrium state for the system $([2], F, \Psi_1)$ since*

$$\int \Psi_1 \, d\mu_{\Psi_1} = -\infty.$$

*This follows since*

$$\int \Psi_1 \, d\mu_{\Psi_1} = \sum_n (s_n - np_\varphi(t)) e^{s_n} \asymp \sum_n \frac{a - \gamma \log n - np_\varphi(t)}{n^\gamma},$$

*so for all large $n$ the summands are dominated by the terms $-p_\varphi(t) n^{1-\gamma}$ which are not summable.*

**Remark 6.5.** *If we wanted the limit of $\mu_t$ as $t \to \infty$ to be a measure with positive entropy, then one way would be to choose our dynamics to be $x \mapsto 5x \mod 1$ and the set $M_0$ to correspond to the interval $[0, 2/5]$ for example.*

Note that for our examples, we can not produce more than two equilibrium states simultaneously. One can see this as following since we are essentially working with two intermingled systems.

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## References

[Aa] J. Aaronson, *An introduction to infinite ergodic theory,* Mathematical Surveys and Monographs, 50. American Mathematical Society, Providence, RI, 1997.

[A] L.M. Abramov, *On the entropy of a flow,* Dokl. Akad. Nauk SSSR **128** (1959) 873–875.

[Ba] L. Barreira, *Dimension and recurrence in hyperbolic dynamics.* Progress in Mathematics, 272. Birkhäuser Verlag, Basel, 2008.

[Bo1] R. Bowen, *Symbolic dynamics for hyperbolic flows*, Amer. J. Math. **95** (1973) 429–460.

[Bo2] R. Bowen, *Some systems with unique equilibrium states,* Math. Systems Theory **8** (1974) 193–202.

[Bo3] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Second revised edition. With a preface by David Ruelle. Edited by Jean-René Chazottes. Lecture Notes in Mathematics, 470. Springer-Verlag, Berlin, 2008.

[Br] H. Bruin, *Minimal Cantor systems and unimodal maps,* J. Difference Equ. Appl. **9** (2003) 305–318.

[BT1] H. Bruin, M. Todd, *Equilibrium states for potentials with* $\sup\varphi - \inf\varphi < h_{top}(f)$*,* Comm. Math. Phys. **283** (2008) 579–611.

[BT2] H. Bruin, M. Todd, *Equilibrium states for interval maps: the potential* $-t\log|Df|$*,* Ann. Sci. Éc. Norm. Supér. **42** (2009) 559–600.

[ChH] J.R. Chazottes, M. Hochman, *On the zero-temperature limit of Gibbs states,* Comm. Math. Phys. **297** (2010) 265–281.

[CRL] M.I. Cortez, J. Rivera-Letelier, *Invariant measures of minimal post-critical sets of logistic maps,* Israel J. Math. 176 (2010) 157–193.

[C1] V. Cyr, *Transient Markov Shifts* Ph.D. thesis, The Pennsylvania State University, 2010.

[C2] V. Cyr, *Countable Markov shifts with Transient Potentials,* Proc. London Math. Soc. **103** (2011) 923–949.

[CS] V. Cyr, O. Sarig, *Spectral Gap and Transience for Ruelle Operators on Countable Markov Shifts,* Comm. Math. Phys. **292** (2009) 637–666.

[D] N. Dobbs, *Renormalisation induced phase transitions for unimodal maps,* Comm. Math. Phys. **286** (2009) 377–387.

[Fe] W. Feller, *An introduction to probability theory and its applications.* Vol. I. Third edition John Wiley and Sons, Inc., New York-London-Sydney 1968.

[FL] A.M. Fisher, A. Lopes, *Exact bounds for the polynomial decay of correlation,* $1/f$ *noise and the CLT for the equilibrium state of a non-Hölder potential,* Nonlinearity **14** (2001) 1071–1104,.

[Gu1] B.M. Gurevič, *Topological entropy for denumerable Markov chains,* Dokl. Akad. Nauk SSSR **10** (1969) 911–915.

[Gu2] B.M. Gurevič, *Shift entropy and Markov measures in the path space of a denumerable graph,* Dokl. Akad. Nauk SSSR **11** (1970) 744–747.

[H] F. Hofbauer, *Examples for the nonuniqueness of the equilibrium state,* Trans. Amer. Math. Soc. **228** (1977) 223–241.

[HK] F. Hofbauer, G. Keller, *Equilibrium states for piecewise monotonic transformations,* Ergodic Theory Dynam. Systems **2** (1982) 23–43.

[IT1] G. Iommi, M. Todd, *Natural equilibrium states for multimodal maps,* Comm. Math. Phys. **300** (2010) 65–94.

[LSV] C. Liverani, B. Saussol, S. Vaienti, *Probabilistic approach to intermittency,* Ergodic Theory Dynam. Systems **19** (1999) 671–685.

[Lo] A.O. Lopes, *The zeta function, nondifferentiability of pressure, and the critical exponent of transition,* Adv. Math. **101** (1993) 133–165.

[MP] P. Manneville, Y. Pomeau, *Intermittent transition to turbulence in dissipative dynamical systems,* Comm. Math. Phys. **74** (1980) 189–197.

[MPa] N.G. Markley, M.E. Paul, *Equilibrium states of grid functions,* Trans. Amer. Math. Soc. **274** (1982) 169–191.

[MU1] R. Mauldin, M. Urbański, *Dimensions and measures in infinite iterated function systems,* Proc. London Math. Soc. (3) **73** (1996) 105–154.

[O] E. Olivier, *Multifractal analysis in symbolic dynamics and distribution of pointwise dimension for g-measures,* Nonlinearity **12** (1999) 1571–1585.

[PP] W. Parry, M. Pollicott, *Zeta Functions and the Periodic Orbit Structure of Hyperbolic Dynamics*, Astérisque 187–188, 1990.

[PS] Y. Pesin, S. Senti, *Equilibrium measures for maps with inducing schemes,* J. Mod. Dyn. **2** (2008) 1–31.

[Pi] V. Pinheiro, *Expanding Measures,* Ann. Inst. H. Poincaré Anal. Non Linéaire **28** (2011) 889–939.

[PU] F. Przytycki, M. Urbański, *Conformal Fractals: Ergodic Theory Methods,* Cambridge University Press 2010.

[Ra] M. Ratner, *Markov partitions for Anosov flows on $n$-dimensional manifolds*, Israel J. Math. **15** (1973), 92–114.

[Ru1] D. Ruelle, *Statistical mechanics on a compact set with $Z^v$ action satisfying expansiveness and specification,* Trans. Amer. Math. Soc. 187 (1973) 237–251.

[S1] O. Sarig, *Thermodynamic formalism for countable Markov shifts,* Ergodic Theory Dynam. Systems **19** (1999) 1565–1593.

[S2] O. Sarig, *Thermodynamic formalism for null recurrent potentials,* Israel J. Math. **121** (2001) 285–311.

[S3] O. Sarig, *Phase transitions for countable Markov shifts,* Comm. Math. Phys. **217** (2001) 555–577.

[S4] O. Sarig, *Lectures Notes on Thermodynamic Formalism for Topological Markov Shifts* (2009) (`http://www.wisdom.weizmann.ac.il//~sarigo/TDFnotes.pdf`).

[S5] O. Sarig, *Symbolic dynamics for surface diffeomorphisms with positive topological entropy* Preprint (arXiv:1105.1650).

[T] M. Todd, *Multifractal analysis for multimodal maps,* Preprint.

[V1] D. Vere-Jones, *Geometric ergodicity in denumerable Markov chains,* Quart. J. Math. Oxford Ser. (2) **13** (1962) 7–28.

[V2] D. Vere-Jones, *Ergodic properties of nonnegative matrices I,* Pacific J. Math. **22** (1967) 361–386.

[W1] P. Walters, *Ruelle's operator theorem and g-measures,* Trans. Amer. Math. Soc. 214 (1975) 375–387.

[W2] P. Walters, *Equilibrium states for $\beta$-transformations and related transformations,* Math. Z. **159** (1978) 65–88.

[W3] P. Walters, *Invariant measures and equilibrium states for some mappings which expand distances,* Trans. Amer. Math. Soc. **236** (1978) 121–153.

[W4] P. Walters, *Convergence of the Ruelle operator for a function satisfying Bowen's condition,* Trans. Amer. Math. Soc. **353** (2001) 327–347.

[W5] P. Walters, *A natural space of functions for the Ruelle operator theorem,* Ergodic Theory Dynam. Systems **27** (2007) 1323–1348.

[Z] R. Zweimuller, *Invariant measures for general(ized) induced transformations,* Proc. Amer. Math. Soc. **133** (2005) 2283–2295.

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