EQUILIBRIUM STATES FOR INTERVAL MAPS: POTENTIALS WITH $\sup\varphi - \inf\varphi < h_{top}(f)$

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Abstract. We study an inducing scheme approach for smooth interval maps to prove existence and uniqueness of equilibrium states for potentials φ with the 'bounded range' condition $\sup\varphi - \inf\varphi < h_{top}(f)$, first used by Hofbauer and Keller [HK]. We compare our results to Hofbauer and Keller's use of Perron-Frobenius operators. We demonstrate that this 'bounded range' condition on the potential is important even if the potential is Hölder continuous. We also prove analyticity of the pressure in this context.

1. Introduction

Thermodynamic formalism is concerned with existence and uniqueness of measures μ_φ that maximise the *free energy*, *i.e.,* the sum of the entropy and the integral over the potential. In other words

$$h_{\mu_\varphi}(f) + \int_X \varphi\, d\mu_\varphi = P(\varphi) := \sup_{\nu\in\mathcal{M}_{erg}} \left\{ h_\nu(f) + \int_X \varphi\, d\nu : -\int_X \varphi\, d\nu < \infty \right\}$$

where $\mathcal{M}_{erg}$ is the set of all ergodic f-invariant Borel probability measures. Such measures are called *equilibrium states*, and $P(\varphi)$ is the *pressure*. This theory was developed by Sinai, Ruelle and Bowen [Si, R, Bo] in the context of Hölder potentials on hyperbolic dynamical systems, and has been applied to Axiom A systems, Anosov diffeomorphisms and other systems too, see e.g. [Ba, K2] for more recent expositions.

In this paper we are interested in smooth interval maps $f : I \to I$ with a finite number of critical points. More precisely, $\mathcal{H}$ will be the collection of *topologically mixing* (*i.e.,* for each $n \geqslant 1$, f^n has a dense orbit) C^2 maps on the interval (or circle) such that all its periodic points are hyperbolically repelling and all its critical points are non-flat. The existence of critical points prevents such maps from being uniformly hyperbolic for the 'natural' potential $\varphi = -\log|Df|$.

Inducing schemes where used in [PeSe] to regain hyperbolicity and prove the existences of equilibrium states for $-t\log|Df|$ for a large interval of t, but very specific Collet-Eckmann unimodal maps f. In [BrT] we investigated $-t\log|Df|$ with t close

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to 1 for multimodal maps whose derivatives critical orbits satisfy only polynomial growth. Combining inducing schemes with ideas of so-called Hofbauer towers and infinite state Markov chains (as presented by Sarig [Sa1, Sa2, Sa3]), we proved the existence and uniqueness of equilibrium states within the class

$$\mathcal{M}_+ = \{\mu \in \mathcal{M}_{erg} : \lambda(\mu) > 0, \mathrm{supp}(\mu) \not\subset \mathrm{orb}(\mathrm{Crit})\}.$$

where $\lambda(\mu) = \int \log |Df| d\mu$ is the Lyapunov exponent of μ. In fact the assumptions that we make on the potentials in this paper ensure that any equilibrium state must lie in this class, and hence it is no restriction to only consider measures there.

Remark 1. *Note that the function $\mu \mapsto h_\mu(f)$ is upper semicontinuous, cf.* [BrK, Lemma 2.3]*. Hence, if the potential is upper semicontinuous, then the free energy map $\mu \mapsto h_\mu(f) + \int \varphi \, d\mu$ is upper semicontinuous too. As $\mathcal{M}_{erg}$ is compact in the weak topology, this gives the existence of equilibrium states, but not uniqueness.*

In this work we want to use inducing schemes to prove existence and uniqueness of equilibrium states for "general" potentials. In this area, there are many results, in particular several papers by Hofbauer and Keller [H1, H2, HK] from the late 1970s. These results were inspired by Bowen's exposition [Bo] for hyperbolic dynamical systems, and investigate what happens when hyperbolicity fails. Their main tool was the Perron-Frobenius operator, which even for non-uniformly expanding interval maps continues to have a quasi-compact structure for many potentials. In this paper we focus on what can be proved for these problems using inducing techniques. We then apply Sarig's theory of countable Markov shifts. (A related application of that theory for multidimensional piecewise expanding maps can be found in [BuSa].) In [HK] two main sets of results are given, based on different regularity conditions for the potential; we will present them briefly in Sections 1.1 and 1.2. At the same time we set out some definitions which will be used throughout the paper. In Section 1.4 we present our main results.

1.1. **Potentials in BV.** Given a function $\varphi : I \to \mathbb{R}$, we define the semi-norm $\|\cdot\|_{BV}$ as

$$\|\varphi\|_{BV} := \sup_{N \in \mathbb{N}} \sup_{0=a_0<\cdots<a_N=1} \sum_{k=0}^{N-1} |\varphi(a_{k+1}) - \varphi(a_k)|.$$

We say that $\varphi \in BV$ if $\|\varphi\|_{BV} < \infty$.

The following result is proved by Hofbauer and Keller in [HK].

Theorem 1 (Hofbauer and Keller)**.** *Let $f \in \mathcal{H}$ and $\varphi \in BV$. If*

$$\sup \varphi - \inf \varphi < h_{top}(f), \tag{1}$$

then there exists an equilibrium state for φ. Moreover, the transfer operator defined by

$$\mathcal{L}_\varphi g(x) := \sum_{y \in f^{-1}(x)} e^{\varphi(y)} g(y)$$

is quasi-compact.

Condition (1) stipulates that φ does not vary too much; similar conditions have been used by e.g. Denker and Urbański [DU] for rational maps on the Riemann sphere, and by Oliveira [O] for higher dimensional maps without critical points. We next state a similar result to Theorem 1 from [DKU, P]. Paccaut [P] also gives many interesting statistical properties for the equilibrium states.

Theorem 2 (Paccaut)**.** *Suppose that φ satisfies*

(a) $\exp(\varphi) \in BV$*;*
(b) $\sum_{n=1}^{\infty} \sup_{C\in\mathcal{P}_n} \|\varphi|_C\|_{BV} < \infty$*;*
(c) $\sup\varphi < P(\varphi)$.

Then there exists a unique equilibrium state μ_φ for φ.

Note that condition (b) on φ is stronger than the condition $\varphi \in BV$, used in Theorem 1. It is also stronger than that in our results in Section 1.4. However, (1) implies condition (c). This follows since assuming (1), the measure of maximal entropy $\mu_{h_{top}(f)}$ gives

$$P(\varphi) \geqslant h_{top}(f) + \int \varphi \, d\mu_{h_{top}(f)} \geqslant h_{top}(f) + \inf\varphi > \sup\varphi.$$

Condition (c) implies that any equilibrium state μ must have $h_\mu(f) \geqslant P(\varphi) - \sup\varphi > 0$. Similarly, supposing (1), and using Ruelle's inequality on Lyapunov exponents (*i.e.,* $h_\mu(f) \leqslant \lambda(\mu)$, see [Ru1]), equilibrium states μ satisfy

$$\begin{aligned} \lambda(\mu) &\geqslant h_\mu(f) = P(\varphi) - \int \varphi \, d\mu \\ &\geqslant h_{top}(f) + \int \varphi \, d\mu_{h_{top}(f)} - \sup\varphi \geqslant h_{top}(f) - (\sup\varphi - \inf\varphi) > 0. \quad (2) \end{aligned}$$

Hence $P_+(\varphi) := \sup_{\mu\in\mathcal{M}_+}\{h_\mu(f) + \int \varphi \, d\mu\} = P(\varphi)$, unless the equilibrium state is supported on orb(Crit).

1.2. **Potentials with Summable Variations.** The results that we want to present rely on a different approach to variation to that above, which is closer to symbolic dynamics. Let $\mathcal{P}_1$ be the partition of I into maximal interval of monotonicity (the *branch partition*) and write $\mathcal{P}_n = \bigvee_{i=0}^{n-1} f^{-i}(\mathcal{P}_1)$. With respect to this partition we define that *n-th variation*

$$V_n(\varphi) := \sup_{\mathbf{C}_n\in\mathcal{P}_n} \sup_{x,y\in\mathbf{C}_n} |\varphi(x) - \varphi(y)|,$$

In this context the following was proved in [HK].

Theorem 3 (Hofbauer and Keller)**.** *Let $f \in \mathcal{H}$ be C^3 and let φ be a potential so that*

(i) *it has summable variations,* i.e., $\sum_n V_n(\varphi) < \infty$*;*
(ii) *the following specification-like property holds: for every $x \in I$, there is k and an increasing sequence $\{n_i\}_i$ such that*

$$\cup_{j=1}^{k} f^{n_i+j}(\mathbf{C}_{n_i}[x]) = I,$$

where $\mathbf{C}_{n_i}[x] \in \mathcal{P}_{n_i}$ is the n_i-cylinder containing x.

Then there exists an equilibrium state for φ and the transfer operator $\mathcal{L}_\varphi$ is quasi-compact.

Property (ii) above is not automatic for interval maps, and it is stronger than the standard specification property which holds for all topologically transitive interval maps, see [Bl] and [Bu1]. For instance, the Fibonacci unimodal map, or more generally, every map with a persistently recurrent critical point (see e.g. [Br2]) fails this condition. In [DKU], Denker et al. replace the conditions of Theorem 3 to (i) $P(\varphi) > \sup \varphi$ and (ii) $\sup_n \beta_n(\varphi) < \infty$, where β_n is defined in (5).

Notice that the set of potentials with summable variations and the set BV have non-empty intersection, but neither is contained in the other, as the following examples demonstrate.

Example 1: Let $f(x) = 2x \pmod 1$ on $[0,1]$ be the doubling map. Clearly, the n-cylinders of f are dyadic intervals of length 2^{-n}. The potential function

$$\varphi(x) := \begin{cases} 0 & \text{if } x = 0; \\ \frac{-1}{\log x} & \text{if } x \in (0, \frac{1}{2}); \\ \frac{1}{\log 2} & \text{if } x \in [\frac{1}{2}, 1], \end{cases}$$

is increasing and bounded, and has $\|\varphi\|_{BV} = \frac{1}{\log 2}$. However, $V_n(\varphi) \geqslant \frac{1}{n \log 2}$, because $\varphi(2^{-n}) - \varphi(0) = \frac{1}{n \log 2}$. So $\sum_n V_n(\varphi)$ diverges. Note that φ is not Hölder either.

Example 2: For f as in Example 1, the potential function

$$\psi(x) := \sum_{n \geqslant 1} \psi_n(x), \text{ where } \psi_n(x) := 4^{-n} \sin(4^{n+1}\pi x) \cdot 1_{\left[\frac{1}{2^n}, \frac{1}{2^{n-1}}\right]}(x)$$

has $\|\psi\|_{BV} = \sum_n \|\psi_n\|_{BV} = \infty$ since $\|\psi_n\|_{BV} = 2$. But $V_n(\psi) \leqslant 4 \cdot 2^{-n}$, so it has summable variations. Note that this function is Lipschitz.

1.3. **Lifting Potentials to Inducing Schemes.** An inducing scheme (X, F, τ) over (I, f) consists of an interval $X \subset I$ containing a (countable) collection of disjoint subintervals X_i, and inducing time $\tau : X \to \mathbb{N}$ such that $\tau_i := \tau|_{X_i}$ is constant and $F|_{X_i} := f^{\tau_i}|_{X_i}$ is monotone onto X. If μ_F is an F-invariant measure, and $\int_X \tau d\mu_F < \infty$, then μ_F can be projected to an f-invariant measure μ as in formula (3) below. Any measure μ that can be obtained this way is called compatible to the inducing scheme. See Section 2.1 the precise definitions.

Proposition 1 below gives a general way of constructing inducing schemes, which we will apply throughout the paper. In Section 2.2, we explain the procedure of lifting measures μ to Hofbauer tower $(\hat{I}, \hat{f})$, which is behind the construction in this proposition. The full proof of Proposition 1 is given in [BrT, Theorem 3 and Lemma 2].

Proposition 1. *If $\mu \in \mathcal{M}_+$ then it is compatible to some induced system (X, F, τ) that corresponds to a first return map to a set $\hat{X}$ on the Hofbauer tower, where $\hat{\mu}(\hat{X}) > 0$. So $\frac{1}{\hat{\mu}(\hat{X})} \int_{\hat{X}} \tau \, d\hat{\mu} < \infty$, and in addition, we can take $X \in \mathcal{P}_n$ for some n.*

Conversely, if an inducing scheme (X, F, τ) has a non-atomic F-invariant measure μ_F such that $\int \tau \, d\mu_F < \infty$, then it projects to an f-invariant measure $\mu \in \mathcal{M}_+$.

Given a potential φ, the lifted potential Φ for inducing scheme (X, F, τ) is given by $\Phi(x) := \sum_{k=0}^{\tau(x)-1} \varphi \circ f^k(x)$. If

$$\sum_n V_n(\Phi) < \infty, \tag{SVI}$$

then we say that φ satisfies the summable variations for induced potential condition, with respect to this inducing scheme. Lemmas 3 and 4 give general conditions on φ and/or the inducing scheme that imply (SVI).

1.4. **Main Results.** After these preparation we can state our main results on the existence and uniqueness of equilibrium states, and analyticity of the pressure function. The existence of equilibrium states in $\mathcal{M}_{erg}$ often follows by Remark 1, but the following theorem gives conditions for uniqueness of equilibrium states in $\mathcal{M}_+$.

Theorem 4. *Let $f \in \mathcal{H}$ and φ be a potential such that $\sup\varphi - \inf\varphi < h_{top}(f)$ and $V_n(\varphi) \to 0$. If the induced potentials corresponding to the inducing schemes given by Proposition 1 satisfies* (SVI)*, then*

(a) *there exists a unique equilibrium state μ_φ;*
(b) *μ_φ is compatible to an induced system with inducing time such that the tails $\mu_\Psi(\{\tau > n\})$ decrease exponentially. (Here μ_Ψ is the equilibrium state of the induced potential $\Psi(x) = \sum_{k=j}^{\tau(x)-1} \psi \circ f^j(x)$ of $\psi := \varphi - P(\varphi)$.)*

Note that $V_n(\varphi) \to 0$ implies that φ can only have discontinuities at precritical points.

Remark 2. *If the tails $\mu_\Psi(\{\tau > n\})$ decrease at certain rates, then one can deduce many statistical properties of the equilibrium state. For instance, exponential decay of correlations follows from exponential tails, see* [Y]*, but for the Central Limit Theorem, Invariance Principles, e.g.* [MN1] *and large deviations* [MN2]*, already polynomial tail behaviour suffices.*

Instead of a single potential, thermodynamic formalism makes use of families $t\varphi$ of potentials. The occurrence of phase transitions is related to the smoothness of the pressure function $t \mapsto P(t\varphi)$. Using the technique in [BrT] we derive

Theorem 5. *Let $f \in \mathcal{H}$ and φ as in Theorem 4. Then the map $t \mapsto P(-t\,\varphi)$ is analytic for t in a neighbourhood of $[-1, 1]$.*

We will not supply a proof of the above theorem, since it follows rather easily from [BrT, Theorem 5]. We will focus our attention on the following related theorem dealing with the potential $-t \log |Df|$. This potential is unbounded, except for $t = 0$. We conclude that $t \mapsto P(-t \log |Df|)$ is analytic near $t = 0$, which is somewhat surprising as we do not require any of the summability conditions of the critical orbits of f used in [BrT].

Theorem 6. *Let $f \in \mathcal{H}$. There exist $t_1 < 0 < t_2$ so that the map $t \mapsto P(-t \log |Df|)$ is analytic for $t \in (t_1, t_2)$. In fact, for $t \in (t_1, t_2)$ there exists a unique equilibrium state with respect to the potential $-t \log |Df|$.*

We next make a detailed study of an example by Hofbauer and Keller [HK, pp32-33] which applies ideas from [H1]. They used it to show the importance of the condition (1) for the quasi-compactness of the transfer operator. We use the example to test the restrictions of the inducing scheme methods, and we also show that (1) cannot simply be replaced by Hölder continuity of the potential by proving the following proposition, cf. [Sa2].

Proposition 2. *For $\alpha \in (0, 1)$, consider the Manneville-Pomeau map $f_\alpha : x \mapsto x + x^{1+\alpha} \pmod 1$. For any $b < -\log 2$, there exists a Hölder potential with $\sup \varphi - \inf \varphi = |b|$ and which has the form $\varphi(x) = -2\alpha x^\alpha$ for x close to 0, which has no equilibrium state accessible from an inducing scheme given by Proposition 1.*

The remainder of this paper is organised as follows. In Section 2 we set out our main tools for generating inducing schemes and applying the theory of thermodynamic formalism. Section 3 contains the tail estimates of inducing schemes we use. In Section 4 we prove our main theorem on existence and uniqueness of equilibrium states. In Section 5 we show that a consequence of our results is an analyticity result for the pressure, with respect to the kind of potentials considered in [BrT]. In Section 6 we give examples, including that in Proposition 2, to show where these techniques break down. Finally in Section 7 we discuss the recurrence implied by compactness of the transfer operator, and we present conditions implying the recurrence of the potential φ.

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2. Equilibrium States via Inducing

2.1. **Inducing Schemes.** As in [BrT] we want to construct equilibrium state via inducing schemes. We say that (X, F, τ) is an *inducing scheme* over (I, f) if

- X is an interval[1] containing a (countable) collection of disjoint intervals X_i such that F maps each X_i homeomorphically onto X.

[1] Due to our assumption that f is topological mixing, we can always find a single interval to induce on, but similar theory works for X a finite union of intervals.

- $F|_{X_i} = f^{\tau_i}$ for some $\tau_i \in \mathbb{N} := \{1, 2, 3 \dots\}$.

The function $\tau : \cup_i X_i \to \mathbb{N}$ defined by $\tau(x) = \tau_i$ if $x \in X_i$, is called the *inducing time*. It may happen that $\tau(x)$ is the first return time of x to X, but that is certainly not the general case. Given an inducing scheme (X, F, τ), we say that a measure μ_F is a *lift* of μ if for all μ-measurable subsets $A \subset I$,

$$\mu(A) = \frac{1}{\Lambda_{F,\mu}} \sum_i \sum_{k=0}^{\tau_i - 1} \mu_F(X_i \cap f^{-k}(A)) \quad \text{for} \quad \Lambda_{F,\mu} := \int_X \tau \, d\mu_F. \tag{3}$$

Conversely, given a measure μ_F for (X, F), we say that μ_F *projects* to μ if (3) holds.

Not every inducing scheme is relevant to every invariant measure. Let $X^\infty = \cap_n F^{-n}(\cup_i X_i)$ is the set of points on which all iterates of F are defined. We call a measure μ *compatible* with the inducing scheme if

- $\mu(X) > 0$ and $\mu(X \setminus X^\infty) = 0$, and
- there exists a measure μ_F which projects to μ by (3), and in particular $\Lambda_{F,\mu} < \infty$.

2.2. **The Hofbauer Tower.** Let $\mathcal{P}_n$ be the branch partition for f^n. The canonical Markov extension (commonly called *Hofbauer tower*) is a disjoint union of subintervals $D = f^n(\mathbf{C}_n)$, $\mathbf{C}_n \in \mathcal{P}_n$, called *domains*. Let $\mathcal{D}$ be the collection of all such domains. For completeness, let $\mathcal{P}_0$ denote the partition of I consisting of the single set I, and call $D_0 = f^0(I)$ the *base* of the Hofbauer tower. Then

$$\hat{I} = \sqcup_{n \geqslant 0} \sqcup_{\mathbf{C}_n \in \mathcal{P}_n} f^n(\mathbf{C}_n) / \sim,$$

where $f^n(\mathbf{C}_n) \sim f^m(\mathbf{C}_m)$ if they represent the same interval. Let $\pi : \hat{I} \to I$ be the inclusion map. Points $\hat{x} \in \hat{I}$ can be written as (x, D) if $D \in \mathcal{D}$ is the domain that $\hat{x}$ belongs to and $x = \pi(\hat{x})$. The map $\hat{f} : \hat{I} \to \hat{I}$ is defined as

$$\hat{f}(\hat{x}) = \hat{f}(x, D) = (f(x), D')$$

if there are cylinder sets $\mathbf{C}_n \supset \mathbf{C}_{n+1}$ such that $x \in f^n(\mathbf{C}_{n+1}) \subset f^n(\mathbf{C}_n) = D$ and $D' = f^{n+1}(\mathbf{C}_{n+1})$. In this case, we write $D \to D'$, giving $(\mathcal{D}, \to)$ the structure of a directed graph. It is easy to check that there is a one-to-one correspondence between cylinder sets $\mathbf{C}_n \in \mathcal{P}_n$ and n-paths $D_0 \to \cdots \to D_n$ starting at the base of the Hofbauer tower and ending at some *terminal domain* D_n. If R is the length of the shortest path from the base to D_n, then the *level of* D_n is $\text{level}(D_n) = R$. Let $\hat{I}_R = \sqcup_{\text{level}(D) \leqslant R} D$.

Several of our arguments rely on the fact that the "top" of the infinite graph $(\mathcal{D}, \to)$ generates arbitrarily small entropy. These ideas go back to Keller [K1], see also [Bu2]. It is also worth noting that the main information is contained in a single transitive part of $\hat{I}$.

Lemma 1. *If I is a finite union of intervals, and the multimodal map $f : I \to I$ is transitive, then there is a closed primitive subgraph $(\mathcal{E}, \to)$ of $(\mathcal{D}, \to)$ containing a dense $\hat{f}$-orbit and such that $I = \pi(\cup_{D \in \mathcal{E}} D)$.*

We denote the transitive part of the Hofbauer tower by $\hat{I}_{\text{trans}}$. For details of the proof see [BrT, Lemma 1].

Let $i : I \to D_0$ be the trivial bijection (inclusion) such that $i^{-1} = \pi|_{D_0}$. Given a probability measure μ, let $\hat{\mu}_0 := \mu \circ i^{-1}$, and

$$\hat{\mu}_n := \frac{1}{n} \sum_{k=0}^{n-1} \hat{\mu}_0 \circ \hat{f}^{-k}. \tag{4}$$

We say that μ is *liftable* to $(\hat{I}, \hat{f})$ if there exists a vague accumulation point $\hat{\mu}$ of the sequence $\{\hat{\mu}_n\}_n$ with $\hat{\mu} \not\equiv 0$, see [K1]. The following theorem is essentially proved there, see [BrK] for more details.

Theorem 7. *Suppose that $\mu \in \mathcal{M}_+$. Then $\hat{\mu}$ is an $\hat{f}$-invariant probability measure on $\hat{I}$, and $\hat{\mu} \circ \pi^{-1} = \mu$.*

Conversely, if $\hat{\mu}$ is $\hat{f}$-invariant and non-atomic, then $\lambda(\hat{\mu}) > 0$.

The strategy followed in [BrT] is to take the first return map to appropriate set in the Hofbauer tower of (I, f) and to use the same inducing time for the projected partition on the interval. Saying that an induced system (X, F, τ) corresponds to a first return map $(\hat{X}, \hat{F}, \tau)$ on the Hofbauer tower means that if $\hat{x} \in \hat{X} \subset \hat{I}$, then $\tau \circ \pi$ is the first return time of $\hat{x}$ under $\hat{f}$ to $\hat{X}$.

2.3. **Pressure and Recurrence.** A topological, *i.e.,* measure independent, way to define pressure was presented in [W]; with respect to the branch partition $\mathcal{P}_1$, it is defined as

$$P_{top}(\varphi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathbf{C}_n \in \mathcal{P}_n} \sup_{x \in \mathbf{C}_n} e^{\varphi_n(x)},$$

where $\varphi_n(x) := \sum_{k=0}^{n-1} \varphi \circ f^k(x)$. We say that the Variational Principle holds if $P(\varphi) = P_{top}(\varphi)$. If φ has sufficiently controlled distortion, then the sum of $\sup_{x\in\mathbf{C}_n} e^{\varphi_n(x)}$ over all n-cylinders can be replaced by the sum of $e^{\varphi_n(x)}$ over all n-periodic points, and thus we arrive at the *Gurevich pressure* w.r.t. cylinder set $\mathbf{C} \in \mathcal{P}_1$.

$$P_G(\varphi) := \limsup_{n\to\infty} \frac{1}{n} \log Z_n(\varphi, \mathbf{C}) \quad \text{for} \quad Z_n(\varphi, \mathbf{C}) := \sum_{f^n x = x} e^{\varphi_n(x)} 1_{\mathbf{C}}(x).$$

If (I, f) is topologically mixing and

$$\beta_n(\varphi) := \sup_{\mathbf{C}_n \in \mathcal{P}_n} \sup_{x,y \in \mathbf{C}_n} |\varphi_n(x) - \varphi_n(y)| = o(n), \tag{5}$$

then $P_G(\varphi)$ is independent of the choice of $\mathbf{C} \in \mathcal{P}_1$, as was shown in [FFY].

Since the branch partition is finite, potentials with bounded variations are bounded, and hence their Gurevich pressure is finite. If φ is unbounded above (whence $P_{top}(\varphi) = \infty$) or the number of 1-cylinders is infinite (as may be the case for induced maps F and induced potential Φ), Gurevich pressure proves its usefulness.

Suppose that (I, f, φ) is topologically mixing. For every $\mathbf{C} \in \mathcal{P}_1$ and $n \geqslant 1$, recall that we defined

$$Z_n(\varphi, \mathbf{C}) := \sum_{f^n x = x} e^{\varphi_n(x)} 1_{\mathbf{C}}(x).$$

Let

$$Z_n^*(\varphi, \mathbf{C}) := \sum_{\substack{f^n x = x, \\ f^k x \notin \mathbf{C} \text{ for } 0<k<n}} e^{\varphi_n(x)} 1_{\mathbf{C}}(x).$$

The potential φ is said to be *recurrent* if[2]

$$\sum_n \lambda^{-n} Z_n(\varphi) = \infty \text{ for } \lambda = \exp P_G(\varphi). \tag{6}$$

Moreover, φ is called *positive recurrent* if it is recurrent and $\sum_n n\lambda^{-n} Z_n^*(\varphi) < \infty$.

In some cases we will use the quantity

$$Z_0(\varphi) := \sum_{\mathbf{C} \in \mathcal{P}_1} \sup_{x \in \mathbf{C}} e^{\varphi(x)}. \tag{7}$$

Proposition 1 of [Sa1] implies that if φ has summable variations then for any $\mathbf{C}$, $Z_n(\varphi, \mathbf{C}) = O(Z_0(\varphi)^n)$. Hence $Z_0(\varphi) < \infty$ implies $P_G(\varphi) < \infty$.

Although we do not assume that the potential φ has summable variations, it is important that the induced potential Φ has summable variations, as we want to apply the following result which collects the main theorems of [Sa3]. We give a simplified version of the original result since we assume that each branch of the induced system (X, F) is onto X. We refer to such a system as a *full shift.*

Theorem 8. *If (X, F, Φ) is a full shift and $\sum_{n\geqslant 1} V_n(\Phi) < \infty$, then Φ has an invariant Gibbs measure if and only if $P_G(\Phi) < \infty$. Moreover the Gibbs measure μ_Φ has the following properties.*

(a) *If $h_{\mu_\Phi}(F) < \infty$ or $-\int \Phi d\mu_\Phi < \infty$ then μ_Φ is the unique equilibrium state (in particular, $P(\Phi) = h_{\mu_\Phi}(F) + \int_X \Phi \, d\mu_\Phi$);*
(b) *The Variational Principle holds,* i.e., *$P_G(\Phi) = P(\Phi)$.*

Note that an F-invariant measure μ is a *Gibbs measure* w.r.t. potential Φ if there is $K \geqslant 1$ such that for every $n \geqslant 1$, every n-cylinder set $\mathbf{C}_n$ and every $x \in \mathbf{C}_n$

$$\frac{1}{K} \leqslant \frac{\mu(\mathbf{C}_n)}{e^{\Phi_n(x) - nP_G(\Phi)}} \leqslant K.$$

Using this theory, the following was proved in [BrT].

Proposition 3. *Suppose that ψ is a potential with $P_G(\psi) = 0$. Let $\hat{X}$ be the set used Proposition 1 to construct the corresponding inducing scheme (X, F, τ). Suppose that the lifted potential Ψ has $P_G(\Psi) < \infty$ and $\sum_{n\geqslant 1} V_n(\Psi) < \infty$.*

Consider the assumptions:

[2] The convergence of this series is independent of the cylinder set $\mathbf{C}$, so we suppress it in the notation.

(a) $\sum_i \tau_i e^{\Psi_i} < \infty$ *for* $\Psi_i := \sup_{x \in X_i} \Psi(x)$*;*
(b) *there exists an equilibrium state* $\mu \in \mathcal{M}_+$ *compatible with* (X, F, τ)*;*
(c) *there exist a sequence* $\{\varepsilon_n\}_n \subset \mathbb{R}^-$ *with* $\varepsilon_n \to 0$ *and measures* $\{\mu_n\}_n \subset \mathcal{M}_+$ *such that every* μ_n *is compatible with* (X, F, τ)*,* $h_{\mu_n}(f) + \int \psi \, d\mu_n \geqslant \varepsilon_n$ *and* $P_G(\Psi_{\varepsilon_n}) < \infty$ *for all* n*;*

If any of the following combinations of assumptions holds:

$$\begin{cases} 1. & (a) \text{ and } (b); \\ 2. & (a) \text{ and } (c); \end{cases}$$

then there is a unique equilibrium state μ *for* (I, f, ψ) *among measures* $\mu \in \mathcal{M}_+$ *with* $\hat{\mu}(\hat{X}) > 0$*. Moreover,* μ *is obtained by projecting the equilibrium state* μ_Ψ *of the inducing scheme and we have* $P_G(\Psi) = 0$*.*

In the remaining part of this section, we give some technical results which connect different ways of computing pressure and Gurevich pressure.

We use the following theorem of [FFY] to show the connection between $P_G(\hat{\varphi})$ and $P_+(\varphi)$.

Theorem 9. *If* (Ω, S) *be a transitive Markov shift and* $\psi : \Omega \to \mathbb{R}$ *is a continuous function satisfying* $\beta_n(\psi) = o(n)$ *then* $P_G(\psi) = P(\psi)$*.*

Corollary 1. *If* $\beta_n(\hat{\varphi}) = o(n)$*, and* $\hat{\varphi}$ *is continuous in the symbolic metric on* $(\hat{I}, \hat{f})$ *then* $P_G(\hat{\varphi}) = P_+(\varphi)$*.*

Proof. We show that the system $(\hat{I}_{\text{trans}}, \hat{f}, \hat{\varphi})$ satisfies the conditions of Theorem 9, where $\hat{I}_{\text{trans}}$ is given below Lemma 1. For $\hat{x}, \hat{y} \in \hat{P}$ with $\hat{P} \in \hat{\mathcal{P}}_n$, we have $|\hat{\varphi}_n(\hat{x}) - \hat{\varphi}_n(\hat{y})| = o(n)$, and Theorem 9 implies $P_G(\hat{\varphi}) = P(\hat{\varphi})$.

It remains to show that $P(\hat{\varphi}) = P_+(\varphi)$. By Theorem 7, any measure in $\mathcal{M}_+$ lifts to $\hat{I}$. We also know that a countable-to-one factor map preserves entropy, provided the Borel sets are preserved by lifting, see [DoS]. For similar arguments, see [Bu2]. Suppose that $\{\hat{\mu}_n\}_n$ is a sequence of $\hat{f}$-invariant measures such that $h_{\hat{\mu}_n}(f) + \int \hat{\varphi} \, d\hat{\mu}_n \to P(\hat{\varphi})$ as $n \to \infty$. Then for the projections $\mu_n = \hat{\mu}_n \circ \pi^{-1}$, $h_{\mu_n}(f) + \int \varphi \, d\mu_n \to P(\hat{\varphi})$ also. So $P_+(\varphi) \geqslant P(\hat{\varphi})$. On the other hand, let $\{\mu_n\}_n \subset \mathcal{M}_+$ be a sequence of measures such that $h_{\mu_n}(f) + \int \varphi \, d\mu_n \to P_+(\varphi)$ as $n \to \infty$. Lifting these measures using Theorem 7, we get $h_{\hat{\mu}_n}(f) + \int \hat{\varphi} \, d\hat{\mu}_n \to P_+(\varphi)$, so $P_+(\varphi) \leqslant P(\hat{\varphi})$ as required. □

We next show that Gurevich pressure can be computed from cylinders of any order.

Lemma 2. *Let* (Ω, f) *be a topologically mixing Markov shift. If* $\varphi : \Omega \to \mathbb{R}$ *satisfies* $\beta_n(\varphi) = o(n)$*, then* $P_G(\varphi, \mathbf{C}) = P_G(\varphi, \mathbf{C}')$ *for any two cylinders* $\mathbf{C}, \mathbf{C}'$ *of any order.*

Proof. Denote the Markov partition of $\hat{I}$ into domains D by $\mathcal{D}$. Take $D, D' \in \mathcal{D}$ such that $\mathbf{C} \subset D$ and $\mathbf{C}' \subset D'$. By transitivity, there is a k-path $\mathbf{C} \subset D \to \cdots \to D'$ and a k'-path $\mathbf{C}' \subset D' \to \cdots \to D$. Then for every n-periodic point $x \in \mathbf{C}$, there is a point

$x' \in \mathbf{C}'$ such that $f^{k'}(x') \in \mathbf{C}_n[x]$, the n-cylinder containing x. Therefore $f^{k'+n}(x') \in \mathbf{C}$ and $f^{k'+n+k}(x') \in x'$. It follows that $e^{\varphi_{n+k+k'}(x')} \leqslant e^{\beta_n + (k+k')\sup \varphi} e^{\varphi_n(x)}$, whence

$$Z_n(\varphi, \mathbf{C}) \geqslant e^{-\beta_n - (k+k')\sup \varphi} Z_{n+k+k'}(\varphi, \mathbf{C}').$$

Therefore, using $\beta_n = o(n)$, we obtain for the exponential growth rate $P_G(\varphi, \mathbf{C}) \geqslant \lim_n \frac{\beta_n}{n} + P_G(\varphi, \mathbf{C}') = P_G(\varphi, \mathbf{C}')$. Reversing the roles of $\mathbf{C}$ and $\mathbf{C}'$ yields $P_G(\varphi, \mathbf{C}) = P_G(\varphi, \mathbf{C}')$. □

2.4. **Summable Variations for the Inducing Scheme (SVI).** In this section we give conditions on φ and under which (SVI) holds for the inducing scheme.

Lemma 3. (a) *If*

$$\sum_n n V_n(\varphi) < \infty;$$

then (SVI) *holds with respect to any inducing scheme.*

(b) *Let φ be α-Hölder continuous and let (X, F, τ) be an inducing scheme obtained from Proposition 1 that satisfies*

$$\sup_i \sum_{k=0}^{\tau_i - 1} |f^k(X_i)|^\alpha < \infty, \tag{8}$$

Then (SVI) *holds w.r.t. that inducing scheme.*

Proof. To prove (a), we apply [Sa1, Lemma 3, Part 1]. Note that the results in the chapter of [Sa1] containing this result are valid if (X, F, τ) is a first return map, which is not true for our case. However, from Proposition 1, we constructed (X, F, τ) to be isomorphic to a first return map on the Hofbauer tower, with potential $\hat{\varphi} = \varphi \circ \pi$. Since $\Phi(x) = \sum_{k=0}^{\tau(x)-1} \varphi \circ f^k(x) = \sum_{k=0}^{\tau(x)-1} \hat{\varphi} \circ \hat{f}^k(\hat{x})$ for each $\hat{x} \in \pi^{-1}(x)$, both the original system and the lift to the Hofbauer tower lead to the same induced potential. Therefore [Sa1, Lemma 3, Part 1] does indeed apply.

Now to prove (b), note that $F : \cup_i X_i \to X$ is extendible, $f^{\tau_i - k} : f^k(X_i) \to X$ has bounded distortion for each $0 \leqslant k < \tau_i$. Consequently, also $f^k : X_i \to f^k(X_i)$ has bounded distortion. Suppose that $|\varphi(x) - \varphi(y)| \leqslant C_\varphi |x-y|^\alpha$. Since $\Phi(x) = \sum_{k=0}^{\tau_i - 1} \varphi \circ f^k(x)$ for $x \in X_i$, we get for $x, y \in X_i$.

$$\begin{aligned} |\Phi(x) - \Phi(y)| &\leqslant \sum_{k=0}^{\tau_i - 1} |\varphi \circ f^k(x) - \varphi \circ f^k(y)| \\ &\leqslant \sum_{k=0}^{\tau_i - 1} C_\varphi |f^k(x) - f^k(y)|^\alpha \\ &\leqslant \sum_{k=0}^{\tau_i - 1} C_\varphi K |f^k(X_i)|^\alpha \cdot \left(\frac{|x-y|}{|X_i|} \right)^\alpha \end{aligned}$$

where K is the relevant Koebe constant for F. Thus the condition in (b) implies that the variation $V_1(\Phi)$ is bounded. Because F is uniformly expanding, the diameter of n-cylinders of F decreases exponentially fast, so if x and $y \in X_i$ belong to the same n-cylinder, the above estimate is exponentially small in n, and summability of variations follows. □

The following lemma gives conditions on f, under which condition (b) can be used for Hölder potentials[3]. We say that $c \in \text{Crit}$ has critical order ℓ_c if there is a constant $C \geqslant 1$ such that $\frac{1}{C}|x-c|^{\ell_c} \leqslant |f(x)-f(c)| \leqslant C|x-c|^{\ell_c}$ for all x; f is *non-flat* if $\ell_c < \infty$ for all $c \in \text{Crit}$.

Lemma 4. *Assume that f is a C^3 multimodal map with non-flat critical points, take $\alpha \in (0,1]$ and let $\ell_{\max} := \max\{\ell_c : c \in \text{Crit}\}$. Then there exists $K = K(\#\text{Crit}, \ell_{\max}, \alpha)$ such that if*

$$\liminf_n |Df^n(f(c))| \geqslant K \qquad \textit{for all } c \in \text{Crit},$$

then formula (8) *holds for every inducing scheme obtained as in Proposition 1 on a sufficiently small neighbourhood of* Crit.

Clearly, if $\liminf_n |Df^n(f(c))| \to \infty$ then (8) holds for all values of $\alpha > 0$ simultaneously.

Proof. This proof is a correction after Rivera-Letelier and Shen drew our attention to a dubious part in the earlier proof. We will use Theorem A in [RS] to fix the gap. This theorem, translated to our notation, says that for every $\beta > 0$, there is $K = K(\#\text{Crit}, \ell_{\max}, \beta)$ and $\rho > 0$ such that if $\min_{c \in \text{Crit}} \liminf_n |Df^n(f(c))| \geq K$, then for every interval J of length $|J| < \rho$ each component of $f^{-n}(J)$ has length $\leq n^{-\beta}$. To give one intermediate step, the condition on the derivatives along critical order implies that f satisfies a backward contraction property with constant r, see [BRSS, Theorem 3], which is then used in [RS, Theorem A], assuming $r = r(\beta) > 1$ is sufficiently large, to show that the components of $f^{-n}(J)$ have length $\leq n^{-\beta}$.

To apply this for our case, we take $\beta > 1/\alpha$ and induce on a set X of length $< \rho$ and then for each partition element X_i, the iterate $f^k(X_i)$ is a component of the $\tau_i - k$-th preimage of X, and therefore has length $\leq (\tau_i - k)^{-\beta}$. This gives $\sum_{k=0}^{\tau_i - 1} |f^k(X_i)|^\alpha \leq \sum_{k=1}^{\tau_i} k^{-\alpha\beta} < \infty$ uniformly over all X_i. Hence formula (8) holds. □

3. Tail Estimates for Inducing Schemes

In the following lemma, we let $\hat{X} \subset \hat{I}_{\text{trans}}$ be a cylinder in $\pi^{-1}(\mathcal{P}_N) \vee \mathcal{D}$ compactly contained in its domain. This cylinder set corresponds to an N-path $q \colon D \to \cdots \to D_N$ in $\hat{I}$. The first return map to $\hat{X}$ is the induced system that we will use.

The growth rate of paths in the Hofbauer tower is given by the topological entropy. Clearly, if we remove $\hat{X}$ from the tower, then this rate will decrease: we will denote it by $h^*_{top}(f)$. If $\hat{X}$ is very small, then h^*_{top} is close to $h_{top}(f)$, so (1) implies that $\sup \varphi - \inf \varphi < h^*_{top}$ for $\hat{X}$ sufficiently small. Note that we can in fact take $\hat{X}$ to be

[3] In Lemma 4 we take inducing schemes on a union of intervals. As in Section 2.1, transitivity implies that this result passes to any single sufficiently small interval.

the type of set, a union of domains in $\hat{I}$, considered in [Br1]. We will use this type of domain in Section 5.

Proposition 4. *Suppose that $V_n(\varphi) \to 0$ and let $\hat{\psi} = \hat{\varphi} - P_G(\hat{\varphi}, \hat{X})$. If $\hat{X} \in \hat{\mathcal{P}}_N$ is so small that*

$$\sup\varphi - \inf\varphi < h^*_{top},$$

then there exist $C, \gamma > 0$ such that $Z^*_n(\hat{\psi}, \hat{X}) < C\, e^{-\gamma n}$.

Proof. We will approximate $Z^*_n(\hat{\varphi}, \hat{X})$ by adding the weights $e^{\hat{\varphi}_{n-1}(\hat{x})}$ of all $n-1$-paths from $\hat{f}(\hat{X})$ to $\hat{X}$ in the Hofbauer tower with outgoing arrows from $\hat{X}$ removed. By removing these arrows we ensure that these paths will not visit $\hat{X}$ before step n, so we indeed approximate $Z^*_n(\hat{\varphi}, \hat{X})$ and not $Z_n(\hat{\varphi}, \hat{X})$. In considering $n-1$-paths, we only miss the initial contribution $e^{\hat{\varphi}|_{\hat{X}}}$ in the weight $e^{\hat{\varphi}_n(\hat{x})}$ for $\hat{x} = \hat{f}^n(\hat{x}) \in \hat{X}$, so it will not effect the exponential growth rate $P^*_G(\hat{\varphi}, \hat{X})$ of $Z^*_n(\hat{\varphi}, \hat{X})$. Since $Z^*_n(\hat{\psi}, \hat{X}) = e^{-nP_G(\hat{\varphi}, \hat{X})} Z^*_n(\hat{\varphi}, \hat{X})$, the proposition follows if we can show the strict inequality $P^*_G(\hat{\varphi}, \hat{X}) < P_G(\hat{\varphi}, \hat{X})$.

Remark: It is this strict inequality that is responsible for the discriminant $\mathfrak{D}_F[\varphi]$ in Section 5 being strictly positive.

The rome technique: We will approximate the Hofbauer tower by finite Markov graphs, and use the following general idea of *romes* in transition graphs from Block et al. [BGMY] to estimate $Z^*_n(\hat{\varphi}, \hat{X})$. Let $\mathcal{G}$ be a finite graph where every edge $i \to j$ has a weight $w_{i,j}$, and let $W = (w_{i,j})$ be the corresponding (weighted) transition matrix. More precisely, $w_{i,j}$ is the total weight of all edges $i \to j$, and if there is no edge $i \to j$, then $w_{i,j} = 0$.

A subgraph $\mathcal{R}$ of $\mathcal{G}$ is called a *rome*, if there are no loops in $\mathcal{G} \setminus \mathcal{R}$. A *simple path* p of length $l(p)$ is given by $i = i_0 \to i_1 \to \cdots \to i_{l(p)} = j$, where $i, j \in \mathcal{R}$, but the intermediate vertices belong to $\mathcal{G} \setminus \mathcal{R}$. Let $w(p) = \prod_{k=1}^{l(p)} w_{i_{k-1}, i_k}$ be the weight of p. The *rome matrix* $A_{rome}(x) = (a_{i,j}(x))$, where i, j run over the vertices of $\mathcal{R}$, is given by

$$a_{i,j}(x) = \sum_p w(p) x^{1-l(p)},$$

where the sum runs over all simple paths p as above. (Note that with the convention that $x^0 = 1$ for $x = 0$, $A_{rome}(0)$ reduces to the weighted transition matrix of the rome $\mathcal{R}$.) The result from [BGMY] is that the characteristic polynomial of W is equal to

$$\det(W - xI_W) = (-x)^{\#\mathcal{G} - \#\mathcal{R}} \det(A_{rome}(x) - xI_{rome}), \tag{9}$$

where I_W and I_{rome} are the identity matrices of the appropriate dimensions.

In our proof, we will use k-cylinder sets as vertices in the graph $\mathcal{G}$, and we will take $w(p) = e^{\hat{\varphi}_{l(p)}(x)}$ for some x belonging to the interval in $\hat{I}$ that is represented by the path p.

Choice of the rome: Fix a large integer k. The partition $\hat{\mathcal{P}}_k$ is clearly a Markov partition for the Hofbauer tower, and its dynamics can be expressed by a countable graph $(\hat{\mathcal{P}}_k, \to)$, where $\hat{P} \to \hat{Q}$ for $\hat{P}, \hat{Q} \in \hat{\mathcal{P}}_k$ only if $\hat{f}(\hat{P}) \supset \hat{Q}$. Choose $R \gg k$ (to be determined later). Given a domain D of level R, from all the R-paths starting at D, at most two (namely those corresponding the the outermost R-cylinders in D) avoid $\hat{I}_R$. Any other R-path from D has a shortest subpath $D \to \cdots \to D'$ where both D and $D' \in \hat{I}_R$. Let us call the union of all points in $\hat{I}$ that belong to one of such subpaths the *wig* of $\hat{I}_R$.

The vertices of the rome $\mathcal{R}$ are those cylinder sets $\hat{P} \in \hat{\mathcal{P}}_k$, $\hat{P} \not\subset \hat{X}$, that are either contained in domains $D \in \mathcal{D}$ of level $< R$, or that belong to the wig. We retain all arrows between two vertices in $\mathcal{R}$. Let $A_{\mathcal{R}}$ be the weighted transition matrix of $\mathcal{R}$. For each arrow $\hat{P} \to \hat{Q}$, choose $\hat{x} \in \hat{P}$ such that $\hat{f}(\hat{x}) \in \hat{Q}$, and set $w_{\hat{P},\hat{Q}} = e^{\hat{\varphi}(\hat{x})}$. Let $\rho_{\mathcal{R}}$ be the leading eigenvalue of the weighted transition matrix. The pressure $P_G^*(\hat{\varphi})$ is approximated (with error of order $V_k(\hat{\varphi})$) by $\log \rho_{\mathcal{R}}$.

The graph $(\mathcal{R}, \to)$ is a finite subgraph of the full infinite Markov graph $(\hat{\mathcal{P}}_k, \to)$. We will construct two other finite graphs $(\mathcal{G}_0, \to)$ and $(\mathcal{G}_1, \to)$ both having $\mathcal{R}$ as a rome, and minorising respectively majorising $(\hat{\mathcal{P}}_k, \to)$ in the following sense: For each path in $(\mathcal{G}_0, \to)$, including those passing through $\hat{X}$, we can assign a path in $(\hat{\mathcal{P}}_k, \to)$ of comparable weight, and this assignment can be done injectively. Conversely, for each path in $(\hat{\mathcal{P}}_k, \to)$, except those passing through $\hat{X}$, we can assign a path in $(\mathcal{G}_1, \to)$ of comparable weight, and this assignment can be done injectively.

As $\mathcal{R}$ is a rome to both $\mathcal{G}_0$ and $\mathcal{G}_1$, we can use the rome technique to compare the spectral radii ρ_0 and ρ_1 of their respective weighted transition matrices W_0 and W_1. By the above minoration/majoration property, we can separate $e^{P_G^*(\varphi)}$ from $e^{P_G(\varphi)}$ by ρ_0 and ρ_1, up to a distortion error. By refining the partition of the Hofbauer tower into k-cylinders, *i.e.,* taking k large, whilst maintaining the majoration/minoration property, we can reduce the distortion error (relative to the iterate), and also show that $\rho_0 < \rho_1$. This will prove the strict inequality $P_G^*(\hat{\varphi}) < P_G(\hat{\varphi})$.

The graph $\mathcal{G}_0$: First, to construct $\mathcal{G}_0$, we add the arrows $\hat{P} \to \hat{Q}$ for each $\hat{P} \in \hat{\mathcal{P}}_k \cap \hat{X}$ and $\hat{Q} \in \hat{\mathcal{P}}_k$ such that $\hat{f}(\hat{P}) \supset \hat{Q}$. The weight of this arrow is $e^{\hat{\varphi}(\hat{x})}$ for some chosen $\hat{x} \in \hat{P}$. Let W_0 be the weighted transition matrix of $\mathcal{G}_0$. It follows that its spectral radius is a lower bound for $e^{P_G(\hat{\varphi})}$, up to an error of order $e^{V_k(\hat{\varphi})}$. Furthermore, the number of n-paths in $\mathcal{R}$ is at least $e^{n(h_{top}(f) - \varepsilon_R)}$, where $\varepsilon_R \to 0$ as $R \to \infty$, cf. [H2]. Since each arrow has weight at least $e^{\inf \hat{\varphi}}$, we obtain

$$e^{h_{top}(f) + \inf \hat{\varphi} - \varepsilon_R} \leqslant \rho_0 := \rho(W_0) \leqslant e^{P_G(\hat{\varphi}) + V_k(\hat{\varphi})}. \tag{10}$$

Let L be such that $f^L(\pi(\hat{X})) \supset I$.

Let $v = (v_{\hat{P}})_{\hat{P} \in \mathcal{P}_k}$ be the positive left unit eigenvector corresponding to the leading eigenvalue $\rho_{\mathcal{R}}$ of $A_{\mathcal{R}}$. Recall that for each $R_0 \in \mathbb{N}$ and $D \in \hat{I}$ there are at most two R_0-paths from D leading to domains of level $> R_0$. Each such path corresponds to a subintervals of D adjacent to ∂D, and although this subinterval may consist of many adjacent cylinder sets of $\hat{\mathcal{P}}_k$, $\hat{f}^R$ maps them monotonically onto adjacent cylinder sets of $\hat{P}_{k-R_0}$. Therefore

$$\begin{aligned}
\sum_{\substack{D\in\mathcal{D}\\ \mathrm{level}(D)>R_0}} \sum_{\hat{Q}\in\mathcal{P}_k\cap D} v_{\hat{Q}} &= \frac{1}{\rho_{\mathcal{R}}^{R_0}} \sum_{\mathrm{level}(\hat{Q})>R_0} \sum_{\hat{P}\in\mathcal{P}_k} v_{\hat{P}} (A_{\mathcal{R}}^{R_0})_{\hat{P},\hat{Q}} \\
&\leqslant 2e^{\sup\hat{\varphi}_{R_0}} \rho_{\mathcal{R}}^{-R_0} \sum_{\hat{P}\in\mathcal{P}_k} v_{\hat{P}} \\
&= 2e^{\sup\hat{\varphi}_{R_0} - R_0 P_G(\hat{\varphi})} \leq 2e^{R_0(\sup\hat{\varphi} - \inf\hat{\varphi} - h_{top}(f))},
\end{aligned}$$

independently of k. Since $\sup\hat{\varphi} - \inf\hat{\varphi} - h_{top}(f) < 0$, we can take R_0 so large, independently of k, that for every $x \in I$,

$$\sum_{\substack{\hat{Q}\in\hat{\mathcal{P}}_k,\ \pi(\hat{Q})\ni x\\ \mathrm{level}(\hat{Q})>R_0}} v_{\hat{Q}} < \frac{1}{2}\min\{v_{\hat{Q}} : \hat{Q}\in\mathcal{P}_k\cap\hat{f}^L(\hat{X}), \pi(\hat{Q})\ni x\}. \tag{11}$$

The idea is now to offset all contributions of n-paths starting from level $> R_0$ to $Z_n^*(\hat{\varphi},\hat{X})$ by the contribution of n-paths starting in $\hat{f}^L(\hat{X})$ to $Z_n(\hat{\varphi},\hat{X})$. Let $N \geqslant L$ be such that there is an N-path from $\hat{X}$ to every $\hat{Q}$ of level $\leqslant R_0$. Then

$$\begin{aligned}
\left(vW_0^N\right)_{\hat{Q}} &\geqslant \left(v(A_{\mathcal{R}}^N + e^{-\inf\hat{\varphi}_N}\Delta)\right)_{\hat{Q}} \\
&\geqslant \begin{cases} (\rho_{\mathcal{R}}^N + e^{-\inf\hat{\varphi}_N}\kappa)v_{\hat{Q}} & \text{if } \mathrm{level}(\hat{Q})\leqslant R_0, \\ \rho_{\mathcal{R}}^N v_{\hat{Q}} & \text{if } \mathrm{level}(\hat{Q}) > R_0, \end{cases}
\end{aligned} \tag{12}$$

where $\kappa := \min\{v_{\hat{Q}} : \hat{Q}\in\hat{\mathcal{P}}_k\cap\hat{f}^L(\hat{X})\}/\max v_{\hat{Q}}$, and Δ a nonnegative square matrix with some 1s in the rows corresponding to $\hat{P}\in\hat{\mathcal{P}}_k\cap\hat{f}^L(\hat{X})$ in such a way that the column corresponding to each $\hat{Q}$ with $\mathrm{level}(\hat{Q})\leqslant R_0$ has at least one 1. The fact that $\kappa > 0$ uniformly in the order of cylinder sets k rests on the following claim, which is proved later on:

$$\min_{\hat{Q}\in\hat{\mathcal{P}}_k\cap\hat{f}^L(\hat{X})} v_{\hat{Q}} / \max_{\hat{Q}\in\hat{\mathcal{P}}_k} v_{\hat{Q}} > 0 \qquad \text{uniformly in } R \text{ and } k. \tag{13}$$

By the choice of L, R_0 (see (11)) and N,

$$\sum_{\hat{Q}\in\hat{\mathcal{P}}_k, \mathrm{level}(\hat{Q})>R_0} e^{\hat{\varphi}_N(\hat{Q})} v_{\hat{Q}} (A_{\mathcal{R}}^N)_{\hat{Q},\hat{P}} \leqslant \frac{1}{2} \sum_{\hat{Q}\in\hat{\mathcal{P}}_k\cap\hat{f}^L(\hat{X})} v_{\hat{Q}} (W_0^N)_{\hat{Q},\hat{P}} \tag{14}$$

for each $\hat{P}$ with $\mathrm{level}(\hat{P})\leqslant R_0$. When we apply W_0^N to (12) once more, the components $v_{\hat{Q}}$ with $\mathrm{level}(\hat{Q})\leqslant R_0$ have increased by a factor $\rho_{\mathcal{R}}^N + e^{-\inf\hat{\varphi}_N}\kappa$, whereas by (14), the components $v_{\hat{Q}}$ with $\mathrm{level}(\hat{Q}) > R_0$ combined amount to at most half the weight of the components $v_{\hat{Q}}$ with $\hat{Q}\in\hat{\mathcal{P}}_k\cap\hat{f}^L(\hat{X})$. Therefore, we can generalise (12) inductively to

$$\begin{aligned}
\left(vW_0^{Nm}\right)_{\hat{Q}} &\geqslant \left(v(A_{\mathcal{R}}^N + e^{-\inf\hat{\varphi}_N}\Delta)^m\right)_{\hat{Q}} \\
&\geqslant \begin{cases} (\rho_{\mathcal{R}}^N + \frac{1}{2}e^{-\inf\hat{\varphi}_N}\kappa)^m v_{\hat{Q}} & \text{if } \mathrm{level}(\hat{Q})\leqslant R_0, \\ \rho_{\mathcal{R}}^{Nm} v_{\hat{Q}} & \text{if } \mathrm{level}(\hat{Q}) > R_0, \end{cases}
\end{aligned} \tag{15}$$

for all $m \geqslant 1$. It follows that

$$\begin{aligned}\sum_{\mathrm{level}(\hat{Q})\leqslant R_0} \frac{1}{\rho_0^{Nm}}(\rho_{\mathcal{R}}^N + \frac{1}{2}e^{-\inf\hat{\varphi}_N}\kappa)^m v_{\hat{Q}} &\leqslant \sum_{\mathrm{level}(\hat{Q})\leqslant R_0} \frac{1}{\rho_0^{mN}}\left(vW_0^{mN}\right)_{\hat{Q}} \\ &\to \alpha \sum_{\mathrm{level}(\hat{Q})\leqslant R_0} w_{\hat{Q}}\end{aligned}$$

for some $\alpha < \infty$ and w the left unit eigenvector corresponding to the leading eigenvalue ρ_0 of W_0. This implies that

$$\rho_0 \geqslant (\rho_{\mathcal{R}}^N + \frac{1}{2}e^{-\inf\hat{\varphi}_N}\kappa)^{1/N} \quad \text{whence} \quad \rho_0 > \rho_{\mathcal{R}} + \kappa' \tag{16}$$

for some $\kappa' = \kappa'(\kappa, N, \hat{\varphi}, R_0) > 0$, uniformly in $R \geqslant R_0$ and $k \in \mathbb{N}$.

The graph $\mathcal{G}_1$: For each $\hat{P} \in \hat{\mathcal{P}}_k \cap D$ where D has level R, consider all R-paths $p : \hat{P} \to \cdots \to \hat{Q}$ that avoid $\hat{I}_R$; these are not included in $(\mathcal{R}, \to)$. From each D of level R, there at most 2 such R-paths avoiding $\hat{I}_R$, corresponding to R-cylinders in D. These two R-cylinders are contained in two k-cylinders in D. For each such k-cylinder $\hat{P}$ (*i.e.,* vertex in $(\hat{\mathcal{P}}_k, \to)$), and each $Q \in \mathcal{P}_k \cap f^R(\pi(\hat{P}))$, choose $\hat{Q} \in \hat{P}_k \cap \hat{I}_R$ and attach an artificial R-path with $R-1$ new vertices and a terminal vertex $\hat{Q}$. Assign weight $w(p) = e^{R\sup\hat{\varphi}}$ to this path. Therefore, if f is d-modal, the number of vertices added to $\hat{I}_R$ is therefore no larger that $2d(R-1)$. Call the resulting graph $\mathcal{G}_1$ and W_1 its weighted transition matrix.

Any n-path in the Hofbauer tower that leaves $\hat{I}_R$ for at least R iterates can be mimicked by an n-path following one of the additional R-paths in $\mathcal{G}_1$. But n-orbits visiting $\hat{X}$ are still left out. It follows that this time, the leading eigenvalue estimate exceeds the exponential growth rate of the contributions of all n-periodic orbits in the Hofbauer tower that avoid $\hat{X}$. Since the error of order $e^{V_k(\hat{\varphi})}$ still needs to be taken into account, we get

$$\rho_1 := \rho(W_1) \geqslant e^{P_G^*(\hat{\varphi}) - V_k(\hat{\varphi})}. \tag{17}$$

On the other hand, we can use (9) to deduce that

$$\det(W_1 - xI_{W_1}) = (-x)^{\#\mathcal{G}_1 - \#\mathcal{R}} \det(A_1(x) - xI_{\mathcal{R}}), \tag{18}$$

where the rome matrix $A_1(x)$ equals $A_{\mathcal{R}}$, except for new entries $w_{\hat{P},\hat{Q}} \leqslant e^{R\sup\hat{\varphi}}x^{1-R}$ for the R-path added to the rome. These paths correspond to R-cylinders, at most 2 for each of the d domains of level R, and since $R \geqslant k$, there are at most $2d$ paths with initial vertices $\hat{P} \in \mathcal{P}_k$, each with at most $\#\mathcal{P}_k$ terminal vertices $\hat{Q}$. In other words, $A_1(x) \leqslant A_{\mathcal{R}} + x^{1-R}e^{R\sup\hat{\varphi}}\Delta_1$, where Δ_1 is a square matrix with at most $2d$ non-zero rows (corresponding to initial vertices $\hat{P}$) and zeros otherwise. Formula (18) shows that ρ_1 is also the leading eigenvalue of $A_1(\rho_1)$.

Although matrices $A_{\mathcal{R}}$ and $\rho_1^{1-R}e^{R\sup\hat{\varphi}}\Delta_0$ depend both on R and k, at the moment we will only need k so large that

$$V_k \leqslant \alpha := \frac{1}{2}\left(h_{top}^*(f) - (\sup\hat{\varphi} - \inf\hat{\varphi})\right) \tag{19}$$

and hence suppress the dependence on k until it is needed again.

We first give some estimates necessary to apply Lemma 6 below with $U_R = A_{\mathcal{R}}$ and $V_R = \rho_1^{1-R} e^{R\sup\hat\varphi}\Delta_1$. The 'left' matrix norm (which is the maximal row-sum) of Δ_1 is $\|\Delta_1\| := \sup_{\|v\|_1=1}\|v\Delta_1\|_1 = \#\mathcal{P}_k$, and therefore (using also (17)) we obtain

$$\begin{aligned}\|\rho^{1-R}e^{\sup\hat\varphi}\Delta_1\| &\leqslant \#\mathcal{P}_k\rho_1^{1-R}e^{R\sup\hat\varphi}\\ &\leqslant \#\mathcal{P}_k e^{R(\sup\hat\varphi - P_G^*(\hat\varphi)+V_k(\hat\varphi))}\\ &\leqslant \#\mathcal{P}_k e^{R(\sup\hat\varphi-\inf\hat\varphi-h^*_{top}(f)+V_k(\hat\varphi))} \leqslant \#\mathcal{P}_k e^{-\alpha R}\end{aligned}$$

for $\alpha > V_k(\hat\varphi)$ as in (19). The entries of $(A^m_{\mathcal{R}})_{\hat P,\hat Q}$ indicate the sum of the weights of all m-paths from $\hat P$ to $\hat Q$. For each $\hat Q \in \hat P_k$, the sum

$$\sum_{\pi(\hat Q)\subset Q}\ \sum_{\text{paths }\hat P\to\hat Q} e^{\sup\hat\varphi_m|_{\hat P}} \leqslant \#\{\text{components of }\pi(\hat P)\cap f^{-m}(Q)\}\ e^{\sup\varphi_m|_{\pi(\hat P)}},$$

which has exponential growth-rate $\rho_{\mathcal{R}}$. Therefore the left matrix norm $\|A^m_{\mathcal{R}}\| \leq \rho^m_{\mathcal{R}}e^{\eta m}$ for some $\eta = \eta(R,k)$ with $\lim_{R\to\infty}\eta(R,k) = 0$ for each fixed k.

If v' is the positive left eigenvector of $A_1(\rho_1)$, corresponding to ρ_1 and normalised so that $\|v'\|_1 := \sum_i |v'_i| = 1$, then

$$\begin{aligned}\rho_1 &= \|v'\rho_1\|_1 = \|v'(A_1(\rho_1)^m\|_1^{1/m}\\ &= \|(A_{\mathcal{R}} + \rho_1^{1-R}e^{R\sup\hat\varphi}\Delta_1)^m\|^{1/m}\\ &\leqslant \rho_{\mathcal{R}}\left(1+\|A_{\mathcal{R}}\|e^{m\tilde\eta(R,k)}\right)^{1/m} \to \rho_{\mathcal{R}}e^{\tilde\eta(R,k)} \qquad \text{as } m\to\infty,\end{aligned} \tag{20}$$

where $\tilde\eta(R,k)$ comes from Lemma 6.

Using (20) and (16) we obtain

$$\rho_1 \leqslant \rho_{\mathcal{R}}e^{\tilde\eta(R,k)} \leqslant e^{\tilde\eta(R,k)}(\rho_0 - \kappa')$$

By claim (13), $\kappa' > 0$ uniformly in R and k, and by Lemma 6, we can choose R large (and hence $\tilde\eta(R,k)$ small) to derive that $\rho_1 < \rho_0$. It follows by (10) and (17) that

$$e^{P_G(\hat\varphi)+V_k(\hat\varphi)} \geqslant \rho_0 > \rho_1 \geqslant e^{P^*_G(\hat\varphi)-V_k(\hat\varphi)}.$$

so taking the limit $k\to\infty$, we get $P_G(\hat\varphi) > P^*_G(\hat\varphi)$ as required.

Proof of Claim (13)**:** We start with the uniformity in R, *i.e.,* the level at which the Hofbauer tower is cut off. Recall that we assumed that $\hat X$ is so small that $\sup\hat\varphi - \inf\hat\varphi < h^*_{top}$. The leading eigenvalue $\rho_{\mathcal{R}}$ of $A_{\mathcal{R}}$ satisfies $\rho_{\mathcal{R}} \geqslant e^{P^*_G(\hat\varphi)-V_k(\hat\varphi)-\varepsilon_R}$ (see (10)), because $\mathcal{R}$ is a subgraph of the Hofbauer tower with $\hat X$ removed. For any r and any domain $D \in \hat I$, there are at most two r-paths ending outside $\hat I_r$. Therefore if $\hat P, \hat Q \in \hat{\mathcal{P}}_k$ where $\hat Q$ is contained in a domain D of level $\geqslant r$, the $\hat P,\hat Q$-entry of $A^r_{\mathcal{R}}$ is at most $2e^{r\sup\hat\varphi}$. Thus we find for the left eigenvector v

$$\rho^r_{\mathcal{R}}\sum_{\hat Q\in\hat{\mathcal{P}}_k\cap D} v_{\hat Q} = \sum_{\hat Q\in\hat{\mathcal{P}}_k\cap D}(vA^r_{\mathcal{R}})_{\hat Q} \leqslant 2e^{r\sup\hat\varphi}\sum_{\hat P\in\hat{\mathcal{P}}_k} v_{\hat P} \leqslant 2e^{r\sup\hat\varphi}.$$

It follows that

$$\sum_{\hat{Q}\in\hat{\mathcal{P}}_k\cap D} v_{\hat{Q}} \leqslant 2e^{r(\sup\hat{\varphi}-P_G^*(\hat{\varphi})+V_k(\hat{\varphi})+\varepsilon_R))} \leqslant 2e^{r(\sup\hat{\varphi}-\inf\hat{\varphi}-h_{top}^*(f)+V_k(\hat{\varphi})+\varepsilon_R)}$$

is exponentially small in r. There are at most $2d$ domains D of level r, which implies that $\sum_{\mathrm{level}(\hat{Q})>r} v_{\hat{P}}$ is exponentially small in r, and this is independent of $R \geqslant r$, and of how (or whether) the Hofbauer tower is truncated.

Next take r_0 so large that $\sum_{\mathrm{level}(\hat{Q})>r_0} v_{\hat{P}} < \frac{1}{2}$ irrespective of the way the Hofbauer tower is cut, and such that $\hat{X}$ belongs to a transitive subgraph of $\hat{I}_{r_0}$. Therefore there is r_0' such that for every domain D of level$(D) \leqslant r_0$ and every $\hat{Q} \in \hat{\mathcal{P}}_k \cap D$, there is an r_0'-path from $\hat{Q}$ to $\hat{X}$. Hence the $\hat{Q}, \hat{P}$ entry in $A_{\mathcal{R}}^{r_0'}$ is at least $e^{r_0' \inf\hat{\varphi}}$ for every $\hat{P} \in \hat{\mathcal{P}}_k \cap \hat{X}$. Since $v = v(\rho_{\mathcal{R}}^{-1}A_{\mathcal{R}})^{r_0'}$, we find

$$\sum_{\hat{P}\in\hat{\mathcal{P}}_k\cap\hat{X}} v_{\hat{P}} \geqslant \rho_{\mathcal{R}}^{-r_0'} e^{r_0' \inf\hat{\varphi}} \sum_{\hat{Q}\in\hat{\mathcal{P}}_k\cap\hat{I}_{r_0}} v_{\hat{Q}} \geqslant \frac{1}{2}\,\rho_{\mathcal{R}}^{-r_0'} e^{r_0' \inf\hat{\varphi}}$$

independently of $R \geqslant r_0$.

Now we continue with the uniformity in k. This is achieved by analysing the effect of splitting of vertices of the transition graph into new vertices, representing cylinders of higher order. We do this one vertex at the time.

Let W be a weighted transition matrix of a graph $\mathcal{G}$. Given a vertex $g \in \mathcal{G}$, we can represent the 2-paths from g by splitting g as follows (for simplicity, we assume that the first row/column in W represents arrows from/to g):

- If $g \to_{w_{1,b_1}} b_1, g \to_{w_{1,b_2}} b_2, \dots, g \to_{w_{1,b_m}} b_m$ are the outgoing arrows, replace g by m vertices $g_1, \dots, g_m$ with outgoing arrows $g_1 \to_{w_{1,b_1}} b_1, g_2 \to_{w_{1,b_2}} b_2, \dots, g_m \to_{w_{1,b_m}} b_m$ respectively, where w_{1,b_j} represents the weight of the arrow.
- Replace all incoming arrows $c \to_{w_{c,1}} g$ by m arrows $c \to_{w_{c,1}} g_1, c \to_{w_{c,1}} g_2, \dots, c \to_{w_{c,1}} g_m$, all with the same weight.
- If $g \to g$ was an arrow in the old graph, this means that g_1 will now have m outgoing arrows: $g_1 \to_{w_{1,1}} g_1, g_1 \to_{w_{1,1}} g_2, \dots, g_1 \to_{w_{1,1}} g_m$, all with the same weight.

Lemma 5. *If W has leading eigenvalue ρ with left eigenvector $v = (v_1, \dots, v_n)$, then the weighted transition matrix $\tilde{W}$ obtained from the above procedure has again ρ as leading eigenvalue, and the corresponding left eigenvector is $\tilde{v} = (\underbrace{v_1, \dots, v_1}_{m\ \text{times}}, v_2, \dots v_n)$.*

Proof. Write $W = (w_{i,j})$ and assume that $w_{1,1} \neq 0$, and the other non-zero entries in the first row are $w_{1,b_2}, \dots, w_{1,b_m}$. The multiplication $\tilde{v}\tilde{W}$ for the new matrix and eigenvector becomes

$$
(\underbrace{v_1,\dots,v_1}_{m\text{ times}}, v_2,\dots v_n)
\begin{array}{l}
\quad\overbrace{\hphantom{w_{1,1}\ \cdots\ w_{1,1}}}^{m\text{ times}} \\
\left(\begin{array}{ccc|ccccccc}
w_{1,1} & \cdots & w_{1,1} & 0 & \cdots & & & & \cdots & 0 \\
0 & \cdots & 0 & \cdots & 0 & w_{1,b_2} & 0 & & & \\
\vdots & & \vdots & & & & & & & \\
0 & \cdots & 0 & \cdots & & & 0 & w_{1,b_m} & 0 & \cdots \\
\hline
w_{2,1} & \cdots & w_{2,1} & w_{2,2} & \cdots & & & & \cdots & w_{2,n} \\
w_{3,1} & \cdots & w_{3,1} & w_{3,2} & \ddots & & & & & \vdots \\
\vdots & & \vdots & & & & & & & \\
\vdots & & \vdots & \vdots & & & & & \ddots & \vdots \\
w_{n,1} & \cdots & w_{n,1} & w_{n,2} & \cdots & & & & \cdots & w_{n,n}
\end{array}\right)
\end{array}
$$

A direct computation shows that this equals $\rho\tilde{v}$. Since $\tilde{v}$ is positive, it has to belong to the leading eigenvalue, so ρ is the leading eigenvalue of $\tilde{W}$ as well. The proof when $w_{1,1} = 0$ is similar. $\square$

The effect of going from $\hat{\mathcal{P}}_k$ to $\hat{\mathcal{P}}_{k'}$ for $k' > k$ is that by repeatedly applying Lemma 5, the entries $v_{\hat{P}}$ for $\hat{P} \in \hat{\mathcal{P}}_k$ have to be replaced by $\#(\hat{P} \cap \hat{\mathcal{P}}_{k'})$ copies of themselves which, when normalised, leads to the new unit left eigenvector $\tilde{v}$. If $\pi(\hat{P}) \subset \pi(\hat{Q})$, then the number of k'-cylinders in $\hat{P}$ is less than the number of k'-cylinders in $\hat{Q}$. Since I contains a finite number of k-cylinders, there is $C = C(k)$ such that $\#(\hat{P} \cap \mathcal{P}_{k'}) \leqslant C\#(\hat{Q} \cap \mathcal{P}_{k'})$ for all $\hat{P}, \hat{Q} \in \mathcal{P}_k$ and $k' > k$. When passing from $\hat{\mathcal{P}}_k$ to $\hat{\mathcal{P}}_{k'}$, we also need to to adjust the weight $e^{\hat{\varphi}(x)}$ for $x \in \hat{P} \in \hat{\mathcal{P}}_k$ slightly, but this adjustment is exponentially small since $V_{k'}(\hat{\varphi}) \to 0$. It follows that $\min_{\hat{P} \in \hat{\mathcal{P}}_{k'} \cap \hat{f}^L(\hat{X})} v_{\hat{P}} / \max v_{\hat{P}}$ is uniformly bounded away from 0, uniformly in k'. $\square$

We finish this section with the technical result used (20).

Lemma 6. *Let $\{U_n\}_{n\in\mathbb{N}}$, $\{V_n\}_{n\in\mathbb{N}}$ be positive square matrices such that $\rho_n \geqslant 1$ is the leading eigenvalue of U_n. Assume that there exist $M < \infty$, $\tau \in (0,1)$ and a sequence $\{\eta_k\}_{k\in\mathbb{N}}$ with $\eta_k \downarrow 0$ as $k \to \infty$ such that for all n*

$$\|U_n\| \leqslant M, \quad \|U_n^k\| \leqslant \rho_n^k e^{k\eta_k} \quad \textit{and} \quad \|V_n\| \leqslant M\tau^n,$$

Then there exists a different sequence $\{\tilde{\eta}_n\}_{n\in\mathbb{N}}$ with $\tilde{\eta}_n \to 0$ as $n \to \infty$ such that

$$\|(U_n + V_n)^j\| \leqslant (1 + e^{j\tilde{\eta}_n})\rho_n^j.$$

In particular, the leading eigenvalue of $\frac{1}{\rho_n}(U_n + V_n)$ tends to 1 as $n \to \infty$.

Remark 3. *Although this lemma works for any matrix norm, we need it for $\|U\| = \sup_{\|v\|_1 = 1} \|vU\|_1$,* i.e., *the maximal row-sum of U_n. Note that we do not assume that all U_n have the same size (although U_n and V_n have the same size for each n).*

Proof. Note that $U_n + V_n$ is a positive matrix and so its leading eigenvalue is equal to the growth rate $\lim_{j\to\infty} \frac{1}{j} \log \|(U_n + V_n)^j\|$. We have

$$(U_n + V_n)^j = \sum_{|p|+|q|=j} U_n^{p_1} V_n^{q_1} \dots U_n^{p_t} V_n^{q_t},$$

where $p = (p_1, \dots, p_t)$, $q = (q_1, \dots, q_t)$ and $|p| = \sum p_i$ and $|q| = \sum q_i$. More precisely, the sum runs over all $t \in \{1, \dots, \lceil j/2 \rceil\}$ and distinct vectors p, q with $p_i, q_i > 0$ (except that possibly $p_1 = 0$ or $q_t = 0$). Let us split the above sum into two parts.

(i) If $|q| > \varepsilon j$, then each of the above terms can be estimated in norm by

$$\|U_n\|^{|p|} \|V_n\|^{|q|} \leqslant M^j (\tau^n)^{\varepsilon j} = (M\tau^{\varepsilon n})^j.$$

Since there are at most 2^j such terms, this gives

$$\left\| \sum_{\substack{|p|+|q|=j \\ |q|>\varepsilon j}} U_n^{p_1} V_n^{q_1} \dots U_n^{p_t} V_n^{q_t} \right\| \leqslant (2M\tau^{\varepsilon n})^j. \tag{21}$$

(ii) If $(q_1, \dots, q_t)$ satisfies $|q| \leqslant \varepsilon j$, then there are at most $t - 1 \leqslant |q|$ indices i with $p_i \leqslant N$ and at least one index i with $p_i > N$, where $N < 1/(2\varepsilon)$ is to be determined later. The norm of each of these terms can be estimated by $\|U_n^{p_1}\| \cdots \|U_n^{p_t}\| M^{|q|} \tau^{n|q|}$, where the factors

$$\|U_n^{p_i}\| \leqslant \begin{cases} \rho_n^{p_i} e^{\eta_N p_i} & \text{if } p_i > N, \\ M^N & \text{if } p_i \leqslant N. \end{cases}$$

So the product of all these factors is at most $\rho_n^j e^{\eta_N j} M^{\varepsilon j N}$. Using Stirling's formula, we can derive that there are at most

$$\sum_{t=0}^{\lfloor \varepsilon j \rfloor} \binom{j}{t} \leqslant \varepsilon j \binom{j}{\lfloor \varepsilon j \rfloor} \leqslant \sqrt{\varepsilon j} \left(\frac{1}{\varepsilon}\right)^{\varepsilon j} \left(\frac{1}{1-\varepsilon}\right)^{(1-\varepsilon)j} \leq e^{\sqrt{\varepsilon} j}$$

possible terms of this form. Combining all this gives an upper bound of this part of

$$\left\| \sum_{\substack{|p|+|q|=j \\ |q|\leqslant\varepsilon j}} U_n^{p_1} V_n^{q_1} \cdots U_n^{p_t} V_n^{q_t} \right\| \leqslant e^{\sqrt{\varepsilon} j} \rho_n^j e^{\eta_N j} M^{\varepsilon j N} (M\tau^n)^{\varepsilon j}. \tag{22}$$

Adding the estimates of (21) and (22), we get

$$\|(U_n + V_n)^j\| \leqslant (2M\tau^{\varepsilon n})^j + e^{\sqrt{\varepsilon} j} \rho_n^j e^{\eta_N j} M^{\varepsilon N j} (M\tau^n)^{\varepsilon j}.$$

Now take $N = n^{\frac{1}{4}}$ and $\varepsilon = n^{-\frac{1}{2}}$ (so indeed $N < 1/(2\varepsilon)$) and n so large that $M\tau^n \leqslant 2M\tau^{\sqrt{n}} \leqslant 1$. Then we get

$$\|(U_n + V_n)^j\| \leqslant \rho_n^j \left(1 + e^{j(n^{-1/4} + \eta_{n^{1/4}} + n^{-1/4} \log M)}\right).$$

The lemma follows with $\tilde{\eta}_n = (n^{-1/4} + \eta_{n^{1/4}} + n^{-1/4} \log M)$. □

4. Proof of Theorem 4

The following is [BrT, Lemma 3].

Lemma 7. *For every $\varepsilon > 0$, there are $R \in \mathbb{N}$ and $\eta > 0$ such that if $\mu \in \mathcal{M}_{erg}$ has entropy $h_\mu(f) \geqslant \varepsilon$, then μ is liftable to the Hofbauer tower and $\hat\mu(\hat I_R) \geqslant \eta$. Furthermore, there is a set $\hat E$, depending only on ε, such that $\hat\mu(\hat E) > \eta/2$ and $\min_{D\in\mathcal{D}\cap\hat I_R} d(\hat E\cap D, \partial D) > 0$.*

The following lemma will allow us to implement condition (c) in Proposition 3.

Lemma 8. *There exist sequences $\{\varepsilon_n\}_n \subset \mathbb{R}^-$ with $\varepsilon_n \to 0$ and $\{\mu_n\}_n \subset \mathcal{M}_+$ so that $h_{\mu_n}(f) + \int\psi\,d\mu_n \geqslant \varepsilon_n$. Moreover, there exists a domain $\hat X$ compactly contained in some $D \in \mathcal{D}$ so that $\hat\mu_n(\hat X) > 0$.*

Proof. First notice that by the definition of pressure, there must exist sequences $\{\varepsilon_n\}_n \subset \mathbb{R}^-$ with $\varepsilon_n \to 0$ and $\{\mu_n\}_n \subset \mathcal{M}_{erg}$ so that $h_{\mu_n}(f) + \int\psi\,d\mu_n \geqslant \varepsilon_n$. By (2), there exists $\varepsilon > 0$ so that we can choose $h_{\mu_n}(f) > \varepsilon$ and $\{\mu_n\}_n \subset \mathcal{M}_+$. Now by Lemma 7, we can choose $\hat X$ compactly contained in some $D \in \mathcal{D}$ and a subsequence $\{n_k\}_k$ with $\hat\mu_{n_k}(\hat X) > 0$ for all k. □

Proof of Theorem 4. Take $\psi := \varphi - P(\varphi)$. By the remark below (2) and Corollary 1, we have $P(\varphi) = P_+(\varphi) = P_G(\hat\varphi)$. Notice that $V_n(\varphi) \to 0$ implies that $\beta_n(\hat\varphi) = o(n)$ and $\hat\varphi$ is continuous in the symbolic metric on $(\hat I, \hat f)$.

Take $\hat X \subset \hat I_{\text{trans}}$ compactly contained in its domain in the Hofbauer tower and satisfying the statement of Lemma 7. By Proposition 4, there are $C, \eta > 0$ such that $Z_n^*(\hat\psi, \hat X) < Ce^{-\eta n}$.

We denote the first return time to $\hat X$ by $r_{\hat X}$, the first return map to $\hat X$ by $R_{\hat X} := \hat f^{r_{\hat X}}$ and the induced potential by $\hat\Psi := \psi_{r_{\hat X}}$. We will shift these potentials, defining $\psi^S := \psi - S$. Then $\hat\Psi^S = \Psi - Sr_{\hat X}$. Since $P_G(\hat\psi) = 0$ and therefore $Z_n(\hat\psi, \hat X) < e^{o(n)}$, we can estimate Z_0 from (7) for $S > -\eta$ as

$$\begin{aligned} Z_0(\hat\Psi^S) = \sum_n \sum_{r_{\hat X}(x)=n} e^{\hat\psi_n(x) - nS} &\leqslant \sum_n Z_n^*(\hat\psi - S, \hat X) \\ &\leqslant C\sum_n e^{n(-S-\eta)} Z_n(\hat\psi, \hat X) \\ &\leqslant C' \sum_n e^{n(P_G(\hat\psi) - S - \eta) + o(n)} < \infty. \end{aligned}$$

Since $P_G(\hat\psi) = 0$, this implies that $P_G(\hat\Psi^S) < \infty$ for all $S > -\eta$. In fact, it also shows that (a) of Proposition 3 holds. We let $S^* \leqslant -\eta < 0$ be minimal such that $P_G(\hat\Psi^S) < \infty$ for all $S > S^*$.

We can prove precisely the same estimates for the map $F = f^\tau$, where $\tau = r_{\hat X} \circ \pi|_{\hat X}^{-1}$, and the potential $\Phi = \varphi_\tau$. That is, for all $S > S^*$, $P_G(\Psi^S) < \infty$ and (a) of

Proposition 3 holds. By Lemma 8, item (c) of Proposition 3 holds. Therefore, Case 2 of Proposition 3 implies that there exists a unique equilibrium state μ_ψ with $\hat{\mu}_\psi(\hat{X}) > 0$.

To show that μ is the unique equilibrium state over I, we assume that there is another equilibrium state μ'. Let $\hat{\mu}'$ be the corresponding measure on $\hat{I}$ from Theorem 7. We now use the fact that $\hat{\mu}$ is positive on cylinders. This follows firstly by the Gibbs properties of the measures obtained for (X, F, μ), and then by the transitivity of (I, f) and $(\hat{I}_{\text{trans}}, \hat{f})$. Thus there exists some cylinder $\hat{X}'$ in the Hofbauer tower which has $\hat{\mu}(\hat{X}'), \hat{\mu}'(\hat{X}') > 0$.

We can use the above arguments to say that the corresponding inducing scheme (X', F', Ψ') satisfies (a) of Proposition 3. But since μ_ψ is an equilibrium state compatible with (X', F'), also (b) is satisfied. Therefore, Case 1 of Proposition 3 completes the proof of uniqueness.

Finally we note that $\mu_\Psi\{\tau > n\}$ decays exponentially in n, since by the Gibbs property there is $C \geqslant 1$ such that

$$\mu_\Psi(\{\tau > n\}) = \sum_{\tau_i > n} \mu_\Psi(X_i) \leqslant C \sum_{\tau_i > n} e^{\Psi_i} = C \sum_{k \geqslant n} Z_k^*(\psi, \hat{X}).$$

By Proposition 4, the latter quantity decays exponentially, as required. □

5. Analyticity of the Pressure Function

In this section we prove Theorem 6. Throughout, let $\varphi_t = -t \log |Df|$. Let $X \subset I$ and (X, F, τ) be an inducing scheme on X where $F = f^\tau$. As usual we denote the set of domains of the inducing scheme by $\{X_i\}_{i \in \mathbb{N}}$. Define a tower over the inducing scheme as follows (see [Y])

$$\Delta = \bigsqcup_{i \in \mathbb{N}} \bigsqcup_{j=0}^{\tau_i - 1} (X_i, j),$$

with dynamics

$$f_\Delta(x, j) = \begin{cases} (x, j+1) & \text{if } x \in X_i, j < \tau_i - 1; \\ (F(x), 0) & \text{if } x \in X_i, j = \tau_i - 1. \end{cases}$$

For $i \in \mathbb{N}$ and $0 \leqslant j < \tau_i$, let $\Delta_{i,j} := \{(x, j) : x \in X_i\}$ and $\Delta_l := \bigcup_{i \in \mathbb{N}} \Delta_{i,l}$ is called the *l-th floor*. Define the natural projection $\pi_\Delta : \Delta \to X$ by $\pi_\Delta(x, j) = f^j(x)$. Note that (Δ, f_Δ) is a Markov system, and the first return map of f_Δ to the *base* Δ_0 is isomorphic (X, F, τ).

Also, given $\psi : I \to \mathbb{R}$, let $\psi_\Delta : \Delta \to \mathbb{R}$ be defined by $\psi_\Delta(x, j) = \psi(f^j(x))$. Then the induced potential of ψ_Δ to the first return map to Δ_0 is exactly the same as the induced potential of ψ to the inducing scheme (X, F, τ).

The differentiability of the pressure functional can be expressed using directional derivatives $\frac{d}{ds} P_G(\psi + sv)\big|_{s=0}$. For inducing scheme (X, F, τ), let ψ_Δ and v_Δ be the lifted potentials to Δ. Suppose that for $\psi_\Delta : \Delta \to \mathbb{R}$, we have $\beta_n(\psi_\Delta) = o(n)$. We define the set of *directions* with respect to ψ:

$$Dir_F(\psi) := \left\{ \upsilon : \sup_{\mu \in \mathcal{M}_+} \left| \int \upsilon \, d\mu \right| < \infty, \ \beta_n(\upsilon_\Delta) = o(n), \ \sum_{n=2}^{\infty} V_n(\Upsilon) < \infty, \text{ and} \right.$$
$$\left. \exists \varepsilon > 0 \text{ s.t. } P_G(\psi_\Delta + s\upsilon_\Delta) < \infty \ \forall \ s \in (-\varepsilon, \varepsilon) \right\},$$

where Υ is the induced potential of υ. Let $\psi^S := \psi - S$ (and so $\Psi^S = \Psi - S\tau$). Set $p_F^*[\psi] := \inf\{S : P_G(\Psi^S) < \infty\}$.[4] If $p_F^*[\psi] > -\infty$, we define the X-*discriminant* of ψ as

$$\mathfrak{D}_F[\psi] := \sup\{P_G(\Psi^S) : S > p_F^*[\psi]\} \leqslant \infty.$$

Given a dynamical system (X, F), we say that a potential $\Psi : X \to \mathbb{R}$ is *weakly Hölder continuous* if there exist $C, \gamma > 0$ such that

$$V_n(\Psi) \leqslant C\gamma^n \text{ for all } n \geqslant 0. \tag{23}$$

The following is from [BrT, Theorem 5].

Theorem 10. *Let $f \in \mathcal{H}$ be a map with potential $\varphi : I \to (-\infty, \infty]$. Suppose that φ satisfies condition* (5)*. Take $\psi = \varphi - P(\varphi)$. Then $\mathfrak{D}_F[\psi] > 0$ if and only if (X, F, μ_Ψ) has exponential tails.*

We are now ready to prove Theorem 6. In this, and the proofs in the sequel, we write $A_n \asymp B_n$ if $\frac{A_n}{B_n} \to 1$ as $n \to \infty$. We also write $A \asymp_{dis} B$ if there exists a distortion constant $K \in [1, \infty)$ so that $\frac{1}{K}A \leqslant B \leqslant KA$.

Proof of Theorem 6. We fix (X, F) as in Proposition 1. Lemma 5 implies that we have exponential tails for the equilibrium state associated to the constant potential $\psi = -h_{top}(f)$, *i.e.*, there exist C, $\eta > 0$ such that

$$\mu_{-\tau h_{top}(f)}\{\tau_i = n\} \leqslant Ce^{-\eta n}. \tag{24}$$

Hence Theorem 10 implies that we have positive discriminant. We can then apply the arguments of the proof of [BrT, Theorem 5] to show that for $\upsilon \in Dir(-h_{top}(f))$, there exists $\varepsilon > 0$ such that $t \mapsto P(-h_{top}(f) + t\upsilon)$ is analytic.

Therefore, in order to ensure analyticity here we must prove $-\log|Df| \in Dir(-h_{top}(f))$. It follows from [BrT, Lemma 7] that this potential has $\sum_{n=2}^{\infty} V_n(-\log|DF|) < \infty$, and [Pr] gives $\sup_{\mu \in \mathcal{M}_+} \left|\int \log|Df| \, d\mu\right| < \infty$; so it only remains to prove that there exists $\varepsilon > 0$ such that $P_G((-h_{top}(f) - t\log|Df|)_\Delta) < \infty$ for $t \in (-\varepsilon, \varepsilon)$. Since $P_G((-h_{top}(f) - t\log|Df|)_\Delta) \leqslant P_G(-\tau h_{top}(f) - t\log|DF|)$, by Abramovs Theorem it suffices to bound $P_G(-\tau h_{top}(f) - t\log|DF|)$. As in Section 2.3, $Z_0(\Phi) < \infty$ implies $P_G(\Phi) < \infty$. In the following calculation we use the fact that for all $\varepsilon > 0$

[4]Note that we use the opposite sign for $p_F^*[\psi]$ to Sarig.

there exists $C_\varepsilon > 0$ so that $\#\{\tau_i = n\} \leqslant C_\varepsilon e^{n(h_{top}(f)+\varepsilon)}$, see the discussion at (29). For $0 < t < 1$, choose $0 < \varepsilon < \left(\frac{t}{1-t}\right) h_{top}(f)$. Using the Hölder inequality,

$$\begin{aligned}
Z_0(-\tau h_{top}(f) - t\log|DF|) &\asymp_{dis} \sum_n e^{-nh_{top}(f)} \sum_{\tau_i=n} e^{-t\log|DF_i|} \\
&\asymp_{dis} \sum_n e^{-nh_{top}(f)} \sum_{\tau_i=n} |X_i|^t \\
&\leqslant \sum_n e^{-nh_{top}(f)} \left(\sum_{\tau_i=n} |X_i|\right)^t (\#\{\tau_i = n\})^{1-t} \\
&\leqslant C_\varepsilon^{1-t} \sum_n e^{n(-h_{top}(f)+(1-t)(h_{top}(f)+\varepsilon))} \\
&= C_\varepsilon^{1-t} \sum_n e^{n(-th_{top}(f)+(1-t)\varepsilon)} < \infty.
\end{aligned}$$

(For further explanation of these calculations see [BrT, Section 5].)

For $t < 0$, first notice that by the Gibbs property of $\mu_{-\tau h_{top}(f)}$

$$\mu_{-\tau h_{top}(f)}\{\tau = n\} \asymp e^{-nh_{top}(f)} \sum_{\tau_i=n} 1 = e^{-nh_{top}(f)} \#\{\tau_i = n\}.$$

Hence, by (24),

$$e^{-nh_{top}(f)} \#\{\tau_i = n\} \leqslant Ce^{-\eta n}. \tag{25}$$

Since $|X_i| \geqslant |X|e^{-\gamma\tau_i}$ for $\gamma := \log\sup|Df|$, we have

$$\begin{aligned}
Z_0(-\tau h_{top}(f) - t\log|DF|) &\asymp_{dis} \frac{1}{|X|^t} \sum_n e^{-nh_{top}(f)} \sum_{\tau_i=n} |X_i|^t \\
&\leqslant \sum_n [e^{-nh_{top}(f)} \#\{\tau_i = n\}] e^{-\gamma nt} \\
&\leqslant C \sum_n e^{-n(t\gamma+\eta)} < \infty,
\end{aligned}$$

if $t\gamma + \eta > 0$. Hence there exists $\varepsilon > 0$ so that $-\log|Df| \in Dir(-h_{top}(f) - \varepsilon)$.

It remains to show existence and uniqueness of equilibrium states. By (25), we have for $t \geqslant 0$, using the Hölder inequality again,

$$\begin{aligned}
Z_0(-t\log|DF| - \tau P(\varphi_t)) &\asymp_{dis} \sum_n e^{-nP(\varphi_t)} \sum_{\tau_i=n} e^{-t\log|DF_i|} \asymp_{dis} \sum_n e^{-nP(\varphi_t)} \sum_{\tau_i=n} |X_i|^t \\
&\leqslant \sum_n e^{-nP(\varphi_t)} \left(\sum_{\tau_i=n} |X_i|\right)^t \#\{\tau_i = n\}^{1-t} \\
&\leqslant \sum_n [e^{-nh_{top}(f)} \#\{\tau_i = n\}]^{1-t} e^{-n(P(\varphi_t)-(1-t)h_{top}(f))} \\
&\leqslant C \sum_n e^{n((1-t)(h_{top}(f)-\eta)-P(\varphi_t))}.
\end{aligned}$$

Since $P(\varphi_t) \to h_{top}(f)$ as $t \to 0$, for all small t we have $(1-t)(h_{top}(f)-\eta') - P(\varphi_t) < 0$. Hence $Z_0(-t\log|DF| - \tau P(\varphi_t)) < \infty$ for small positive t.

For $t < 0$, we use a similar computation as before:

$$\begin{aligned} Z_0(-t\log|DF| - \tau P(\varphi_t)) &\asymp_{dis} \sum_n e^{-nP(\varphi_t)} \sum_{\tau_i=n} e^{-t\log|DF_i|} \\ &\asymp_{dis} \sum_n e^{-nP(\varphi_t)} \sum_{\tau_i=n} |X_i|^t \\ &\leqslant \sum_n e^{-n(P(\varphi_t)+t\gamma)} \#\{\tau_i = n\} \\ &< C_\varepsilon \sum_n e^{-n(t\gamma + P(\varphi_t) - h_{top}(f) - \varepsilon)}, \end{aligned}$$

where we use the fact that for all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ so that $\#\{\tau_i = n\} \leqslant C_\varepsilon e^{n(h_{top}(f)+\varepsilon)}$. Since $P(\varphi_t) > h_{top}(f)$ we can ensure that $t\gamma + P(\varphi_t) - h_{top}(f) - \varepsilon > 0$ for all t close to zero. Hence $Z_0(-t\log|DF| - \tau P(\varphi_t))$ is finite for all t close enough to zero.

This implies that for t in a neighbourhood of 0, $P_G(-t\log|DF| - \tau P(\varphi_t)) < \infty$. Similarly property (a) of Proposition 3 holds, and thus we can apply Case 2 of that proposition to get existence of an equilibrium state μ. This is the unique equilibrium state among those that can be lifted to (X, F). Following the argument in the proof of Theorem 4, we have that μ is the unique global equilibrium state as required. □

6. Necessity of the Condition $\sup\varphi - \inf\varphi < h_{top}(f)$

In this section we show the importance of the condition (1) for the existence and uniqueness of equilibrium states obtained by inducing methods.

Hofbauer and Keller gave an example, originally in a symbolic setting [H1] and later in the context of the angle doubling map on the circle [HK], which showed that (1) is essential for their results on quasi-compactness of the transfer operator. In Section 6.1, we discuss how that example fits in with our inducing results. The Hofbauer and Keller example uses a non-Hölder potential, so it is natural to ask if is really the lack of Hölder regularity which causes problems in obtaining equilibrium states. In Section 6.2, we provide an example of a family of Hölder continuous potentials which, if a member of the family violates (1), then the equilibrium state is not obtained from any inducing scheme with integrable inducing time.

We note here that these Markov examples are often modelled by the renewal shift, see [Sa2] and [PeZ]. That approach uses a rather different partition to the one we use in this paper, and so does not elucidate our theory. However, the inducing schemes we use and the ones that [Sa2] and [PeZ] get from the renewal shift are the same.

6.1. **Hofbauer and Keller's Example.** As mentioned in Theorem 1, potentials $\varphi \in BV$ satisfying $\sup\varphi - \inf\varphi < h_{top}(f)$ have equilibrium states; in fact Hofbauer and Keller [HK] show that this equilibrium state is absolutely continuous w.r.t. to a φ-conformal measure, and that the transfer operator is quasi-compact. They also present, for the angle doubling map $f(x) = 2x \pmod 1$, a class of potentials φ to show that (1) is essential for these latter properties. This map f was inspired by an example based in [H1] based on the full shift $\sigma : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$, showing that Hölderness of potentials is essential to obtain the results from [Bo].

We demonstrate how this class of examples fits into the framework of our paper. Fix $K \geqslant 0$ and let $b < 0$. Let

$$\varphi = \varphi_{b,K} = \sum_{k=0}^{\infty} a_k \cdot 1_{(2^{-k-1}, 2^{-k}]},$$

where

$$a_k := \begin{cases} b & \text{for } 0 \leqslant k < K, \\ 2\log\left(\frac{k+1}{k+2}\right) & \text{for } k \geqslant K. \end{cases}$$

Also let $s_n = \sum_{k=0}^{n-1} a_k$. Since the Dirac measure δ_0 at the fixed point has free energy $h_{\delta_0}(f) + \varphi(0) = 0$, the pressure $P(\varphi) \geqslant 0$. Figure 1 summarises the results of [H1] and the example in [HK] that are relevant for us.

		Pressure $P(\varphi)$	μ_φ is a Gibbs measure	φ has a unique equilibrium state
$\sum_k e^{s_k} > 1$	$\sum_k a_k < \infty$	$P(\varphi) > 0$	yes	yes
	$\sum_k a_k = \infty$	$P(\varphi) > 0$	no	yes
$\sum_k e^{s_k} = 1$	$\sum_k (k+1)e^{s_k} < \infty$	$P(\varphi) = 0$	no	no
	$\sum_k (k+1)e^{s_k} = \infty$	$P(\varphi) = 0$	no	yes
$\sum_k e^{s_k} < 1$		$P(\varphi) = 0$	no	yes

FIGURE 1. Summary of results in [H1]: Equation (2.6) and Section 5.

Define the inducing scheme (X, F) where $X = (\frac{1}{2}, 1]$ and $F : \bigcup_n X_n \to X$ is the first return map to X where for $n \geqslant 1$, $X_n := \left(\frac{1}{2} + 2^{-n-1}, \frac{1}{2} + 2^{-n}\right]$. Notice that if we denote $X^\infty = \{x : \#\text{orb}(x) \cap X = \infty\}$, then $\mu(X^\infty) = 1$ for every measure in $\mathcal{M}_{erg} \setminus \{\delta_0\}$.

In [HK], it is important that b is chosen so that $-b > h_{top}(f) = \log 2$, but for our case we allow b to vary.

Lemma 9. *For all $K \geqslant 2$ there exists $b_K < -\log 2$ such that*

- *$b > b_K$ implies $P(\varphi_{b,K}) > 0$ and there exists a unique equilibrium state which can be found from (X, F);*
- *$b \leqslant b_K$ implies $P(\varphi_{b,K}) = 0$ and the unique equilibrium state is the Dirac measure δ_0 on 0. This cannot be found from (X, F).*

Moreover, $b_K \to -\log 2$ *as* $K \to \infty$.

Proof. Firstly, we compute

$$s_n = \begin{cases} nb & \text{if } n \leqslant K, \\ Kb + 2\log\left(\prod_{j=K}^{n-1}\left(\frac{j+1}{j+2}\right)\right) = Kb + 2\log\left(\frac{K+1}{n+1}\right) & \text{if } n > K. \end{cases}$$

As in [HK], we can estimate

$$\sum_n e^{s_n} = \sum_{n=1}^{K} e^{nb} + e^{Kb}\sum_{n>K}\left(\frac{K+1}{n+1}\right)^2 < e^b\left(\frac{1-e^{bK}}{1-e^b}\right) + e^{bK}(K+1). \tag{26}$$

For $b < -\log 2$ the first term is strictly less than 1 for all K and the second term tends to zero as $b \to -\infty$. Hence if we fix K, then we can find b_K such that $\sum_n e^{s_n} \leqslant 1$ for $b \leqslant b_K$ (with equality if and only if $b = b_K$), and Figure 1 shows that $P(\varphi) = 0$. Alternatively, by fixing $b < -\log 2$ and taking K large enough we have $P(\varphi) = 0$, and in fact $b_K \to -\log 2$ as $K \to \infty$. A computation similar to (26) shows that $\sum_n (n+1)e^{s_n} \geqslant C\sum_{n>K}(n+1)(\frac{K+1}{n+1})^2$ diverges. Whenever $P(\varphi) = 0$, Figure 1 shows that δ_0 is the unique equilibrium state.

We next show what $P(\varphi) = 0$ or $P(\varphi) > 0$ imply for obtaining the equilibrium state from the inducing scheme. As usual, we set $\psi := \varphi - P_G(\varphi)$. Notice that $V_n(\Psi) = 0$, so clearly we have summable variations. Also notice that for $x \in X_n$,

$$\Psi(x) = s_n - nP(\varphi) \asymp -nP(\varphi) + 2\log\left(\prod_{k=0}^{n-1}\left(\frac{k+1}{k+2}\right)\right) = -nP(\varphi) - 2\log(n+1).$$

Therefore

$$Z_0(\Psi) = \sum_{n=1}^{\infty} e^{\Psi|_{X_n}} \asymp_{dis} \sum_{n=0}^{\infty} e^{-nP(\varphi) - 2\log(n+1)} = \sum_{n=0}^{\infty}\frac{e^{-nP(\varphi)}}{(n+1)^2} < \infty, \tag{27}$$

because $P(\varphi) \geqslant 0$. So as in Section 2.3 this means that $P_G(\Psi) < \infty$. Thus Theorem 8 yields a Gibbs state μ_Ψ. Similarly to the calculation above, we can show from the Gibbs property of μ_Ψ that

$$-\int \Psi \, d\mu_\Psi \asymp_{dis} \sum_{n=0}^{\infty}\frac{e^{-nP(\varphi)}\log(n+1)}{(n+1)^2} < \infty.$$

for $P(\varphi) \geqslant 0$. Therefore, μ_Ψ is an equilibrium state for (X, F). We also have

$$\int \tau \, d\mu_\Psi \asymp_{dis} \sum_{n=1}^{\infty}\frac{ne^{-nP(\varphi)}}{(n+1)^2} \qquad \begin{cases} < \infty & \text{if } P(\varphi) > 0, \\ = \infty & \text{if } P(\varphi) = 0. \end{cases} \tag{28}$$

Therefore if $P(\varphi) = 0$, we cannot project this measure to the original system. □

In the limit $K \to \infty$, the potential is $\varphi(x) = b$ for $x \in (0, 1]$ and $\varphi(0) = 0$. It is easy to see that the same results above hold in this case and that for $\varphi_{-\log 2,\infty}$ the equilibrium states are δ_0 and the measure of maximal entropy.

We briefly summarise the conclusions of this example, in order to clarify how it fits in with the results stated in this paper. We fix $K \geqslant 2$. Since φ is monotone,

$\|\varphi\|_{BV} < \infty$, but $\sum_n \sup_{\mathbf{C}\in\mathcal{P}_n} \|\varphi|_{\mathbf{C}}\|_{BV} = \sum_n V_n(\varphi) = \infty$. Therefore Theorem 2 does not apply for any value of b.

- For $b \leqslant b_K$, we have $P(\varphi) = 0$ but (1) fails, so Theorems 1 and 4 do not apply. However, there exists a unique equilibrium state δ_0 by [H1].
- For $b_K < b \leqslant -\log 2$, we have $P(\varphi) > 0$, but again Theorems 1 and 4 do not apply. However, there exists a unique equilibrium state by [H1]. Moreover, direct computations as in (27) and (28) allow us to use our inducing method and Case 2 of Proposition 3 to show that there exists a unique equilibrium state, which can be obtained from an inducing scheme.
- For $-\log 2 < b < 0$, Theorem 1 applies (since $\|\varphi\|_{BV} < \infty$) and Theorem 4 applies because Ψ is piecewise constant (so (SVI) holds and in fact, Ψ is weakly Hölder continuous, see (23)). Both theorems produce the unique equilibrium state.

In general, inducing schemes are used to improve the hyperbolicity of the map or properties of the potential (e.g. to obtain weak Hölder continuity). For this system (or for the Manneville-Pomeau map of Section 6.2 below), there are inducing schemes that produce the equilibrium state δ_0. For instance, one can take the original map itself, or the 'unnatural' system consisting of the left branch only, as induced system. But to obtain nice properties for map or potential, one has to induce to a domain disjoint from 0, and none of these 'natural' inducing schemes produces δ_0 as equilibrium state.

For $b \leqslant b_K$ we have $\mathfrak{D}_F[\varphi] = 0$, since $P_G(\Phi - S\tau) = \infty$ for all $S < 0$. If φ had summable variations, then the discriminant theorem [Sa2] would imply that φ is not 'strong positive recurrent', but can be either positive recurrent or null recurrent. The fact that we cannot project μ_Ψ appears to suggest that φ is null recurrent. However, since the variations of φ are not summable we are not able to use this theory. However, in the following lemma we make a direct computation to show that indeed φ is null recurrent when $b \leqslant b_K$.

Lemma 10. *Fix $K \geqslant 2$. If $b \leqslant b_K$ then φ is null recurrent.*

Proof. Let $\mathbf{C}_0$ and $\mathbf{C}_1$ the left and right cylinders in $\mathcal{P}_1$. Rather than considering all n-periodic cycles, we will restrict ourselves to special ones, and show that these are sufficient to imply recurrence. For each n there is a cycle $\mathrm{cyc}_n := \{p_n^n, \dots, p_n^1\}$ where $p_n^1 \in X_n$ as defined above, $f(p_n^k) = p_n^{k-1}$ for $n \geqslant k \geqslant 2$ and $f(p_n^1) = p_n^n$ (in fact it is easy to compute $p_k^n = \frac{2^{n-k}}{2^n-1}$). For $x \in \mathrm{cyc}_n$, $\varphi_n(x) = s_n$. This cycle features $n-1$ times in the computation of $Z_n(\varphi, \mathbf{C}_0)$. Hence,

$$Z_n(\varphi, \mathbf{C}_0) \geqslant ne^{s_n} \geqslant (n-1-K)\left[\left(\frac{K}{n+1}\right)^2 \cdot e^{Kb}\right],$$

so $\sum_n Z_n(\varphi, \mathbf{C}_0) \geqslant \sum_n \frac{C}{n} = \infty$. Recalling that $P_G(\varphi) = 0$ for $b \leqslant b_K$, this implies that the potential is recurrent.

Notice that p_n^1 is the only point in cyc_n that belongs to $\mathbf{C}_1$. So using this point and cylinder $\mathbf{C}_1$, the same computation implies that $\sum_n nZ_n^*(\varphi, \mathbf{C}_1) = \infty$, so φ is null recurrent. $\square$

6.2. **The Manneville-Pomeau Map.** The Manneville-Pomeau map $f_\alpha(x) = x + x^{1+\alpha} \pmod 1$ with $\alpha \in (0,1)$ is well-known to have zero entropy equilibrium states for the potential $-t\log|Df_\alpha|$ and appropriate values of t. See [Sa2] for an exposition of this theory and the relevant references. Supposing that $\alpha < \frac{\log 2}{2}$, for $p_1 < p_2 < 1$ and $b < -\log 2$ we will use the potential

$$\varphi(x) = \varphi_{\alpha,p_1,p_2,b}(x) := \begin{cases} -2\alpha x^\alpha & \text{if } x \in [0,p_1], \\ \left(\frac{b+2\alpha p_1^\alpha}{p_2-p_1}\right)(x-p_1) - 2\alpha p_1^\alpha & \text{if } x \in (p_1,p_2], \\ b & \text{if } x \in (p_2,1], \end{cases}$$

as an example to show that (1) is sharp. (Note that φ has the same Hölder exponent as $-\log|Df_\alpha|$.) Since $h_{top}(f) = \log 2$, condition (1) is violated whenever $b \leqslant -\log 2$. It turns out that as soon as this occurs, we can choose α, p_1, p_2 so that no equilibrium state can be achieved from a 'natural' inducing scheme on an interval bounded away from the neutral fixed point 0. Thus (1) is sharp, even when the potential is Hölder.

The conclusion of Proposition 2 proved below is that Hölder regularity of the potential is not sufficient to dispense with the condition (1).

Proof of Proposition 2. We will make a suitable choice for p_1, p_2 later in the proof. Let $y_0 = 1$ and define $y_n \in (0, y_{n-1})$ for $n \geqslant 1$ such that $f_\alpha(y_n) = y_{n-1}$. From the recursive relation $y_n = y_{n+1}(1 + y_{n+1}^\alpha)$ we derive (cf. [dB])

$$\frac{1}{y_n} = \frac{1}{y_{n+1}}(1 + y_{n+1}^\alpha)^{-1} = \frac{1}{y_{n+1}}\left(1 - y_{n+1}^\alpha + y_{n+1}^{2\alpha} + Err(y_{n+1}^{3\alpha})\right),$$

where $|Err(y_{n+1}^{3\alpha})| = O(y_{n+1}^{3\alpha})$. Using $u_n = y_n^{-\alpha}$ this becomes

$$\begin{aligned} u_n &= u_{n+1}\left(1 - \frac{1}{u_{n+1}} + \frac{1}{u_{n+1}^2} + Err\left(\frac{1}{u_{n+1}^3}\right)\right)^\alpha \\ &= u_{n+1}\left(1 - \frac{\alpha}{u_{n+1}} + \frac{\alpha(\alpha+1)}{2u_{n+1}^2} + Err\left(\frac{1}{u_{n+1}^3}\right)\right), \end{aligned}$$

where $\left|Err\left(\frac{1}{u_{n+1}^3}\right)\right| = O\left(\frac{1}{u_{n+1}^3}\right)$. Therefore $u_{n+1} - u_n = \alpha + \frac{\alpha(\alpha+1)}{2}\frac{1}{u_{n+1}} + Err\left(u_{n+1}^{-2}\right)$, and using telescoping series this leads to

$$u_n = \alpha n + \frac{\alpha(\alpha+1)}{2}\log n + Err\left(\frac{1}{n}\right)$$

Transforming back to the original coordinate y_n, we find

$$y_n = \left(\frac{1}{u_n}\right)^{1/\alpha} = \left(\frac{1}{\alpha}\right)^{1/\alpha}\left(\frac{1}{n}\right)^{1/\alpha}\left(1 + \frac{\alpha(\alpha+1)}{2n}\log n + Err\left(\frac{1}{n^2}\right)\right)^{-1/\alpha}.$$

Thus

$$\begin{aligned}\varphi(y_n) = -2\alpha y_n^\alpha &= \frac{-2}{n}\left(1+\frac{\alpha(\alpha+1)}{2n}\log n + Err(n^{-2})\right)^{-1}\\ &= \frac{-2}{n}+\frac{\alpha(\alpha+1)}{n^2}\log n + Err(n^{-3})\end{aligned}$$

for $y_n < p_1$ where $|Err(n^{-3})| = O(n^{-3})$.

For all n sufficiently large, the variations w.r.t. the branch partition satisfy $V_n(\varphi) \geqslant \frac{2}{n}$ (obtained on the n-cylinder set $[0, y_n]$), so φ does not have summable variations. However, since φ is monotone, $\|\varphi\|_{BV} < \infty$.

For any $b < -\log 2$ we will choose $K > N$ and $p_1 = K$ and $p_2 = y_N$ depending on α and b. $n \geqslant N$ implies (replacing the convergent sum of the last given and higher order terms by a single constant $B = B_N$ which is bounded in N),

$$s_n := \sup_{x\in(y_{n+1},y_n]} \sum_{k=0}^{n-1}\varphi(f^k(x)) = Nb - 2\sum_{k=N}^{n-1}\frac{1}{k} + B.$$

Clearly choosing N large enough we can make this error as small as we like. By the above, we have

$$\sum_n e^{s_n} \leqslant \sum_{k=1}^{N} e^{kb} + e^{Nb+B}\sum_{N+1}^{\infty}\left(\frac{N}{n}\right)^2 \leqslant e^b\left(\frac{1-e^{Nb}}{1-e^b}\right) + e^{Nb+B}(N+1).$$

Hence, we can choose N so large that $\sum_n e^{s_n} \leqslant 1$ and hence by Figure 1 we have $P(\varphi) = 0$. (Likewise we can fix suitable α, N, K and find a critical value $b_{\alpha,N,K}$ where below this value, $\sum_n e^{s_n} \leqslant 1$ and above it, $\sum_n e^{s_n} > 1$.)

We define F to be the first return map to $X := (y_1, 1]$, so if $x_i \in (y_1, 1]$ is such that $f_\alpha(x_i) = y_i$, then $X_i = (x_{i+1}, x_i]$ and $\tau_i = i$. A straightforward computation shows that $\Phi|_{X_n}$ is monotone and there is $C \geqslant 1$ such that for large n, $-2\log n - \frac{C}{n} \leqslant \Phi|_{X_n} \leqslant -2\log n + \frac{C}{n}$; in fact Φ is weakly Hölder. As in Lemma 9, we can show that $Z_0(\Phi) < \infty$, so $P_G(\Phi) < \infty$ and there is a unique equilibrium state μ_Φ for (X, F, Φ) which also satisfies the Gibbs property. However, as in (28), the inducing time has $\int \tau \, d\mu_\Phi = \infty$, as required. □

7. Recurrence of Potentials

Although not crucial for the main results of this paper, the question whether the potential is recurrent (see (6)) is of independent interest. In this section we give sufficient conditions for φ to be recurrent, and for the topological pressure and the Gurevich pressure to coincide.

Recall that Theorems 1 and 3 gave conditions under which transfer operator $\mathcal{L}_\varphi$ is quasi-compact. Let us first lay out an argument why this implies that φ is recurrent. Recall that quasi-compactness means that the essential spectrum σ_{ess} is strictly less than the leading eigenvalue $\lambda = \exp(P(\varphi))$, and there are only finitely many eigenvalues outside $\{|z| \leqslant \sigma_{ess}\}$, each with finite multiplicity. A result due to Baladi and Keller [BaK] says that this spectral gap implies that the dynamical ζ-function

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \sum_{f^n(x)=x} e^{\varphi_n(x)}\right)$$

is meromorphic on $\{|z| \leqslant \lambda^{-1}\}$, with a pole at λ^{-1} whose multiplicity is the same as the multiplicity of the eigenvalue λ of $\mathcal{L}_\varphi$. The argument why this implies recurrence of the potential is somewhat implicit in [BaK]. Namely, there is a function g which is analytic on $\{|z| < \lambda^{-1}\} \cap \{|z - \lambda^{-1}| < \varepsilon\}$ such that $g(\lambda^{-1}) \neq 0$ and $\zeta'(z)/\zeta(z) = g(z)/(z-\lambda^{-1})$ on this region. Hence $\lim_{z \to \lambda^{-1}} \zeta'(z)/\zeta(z) = \infty$. Direct computation gives

$$\frac{\zeta'(z)}{\zeta(z)} = \frac{1}{z}\sum_{n=1}^{\infty} z^n \sum_{f^n(x)=x} e^{\varphi_n(x)} = \frac{1}{z}\sum_{n=1}^{\infty} z^n \ Z_n(\varphi),$$

so recurrence follows.

Proposition 5. *Let $f \in \mathcal{H}$ and φ be a potential such that* $\sup\varphi - \inf\varphi < h_{top}(f)$. *If*

$$V_n(\varphi) \to 0 \quad \text{ and } \quad \sum_n e^{-\beta_n} = \infty,$$

then φ is recurrent. (Here $\beta_n := \beta_n(\varphi)$ is defined as in (5)*.)*

Clearly $\beta_n \leqslant \sum_{k=1}^n V_k(\varphi)$, and $V_n(\varphi) \to 0$ implies $\beta_n = o(n)$. The condition $\sum_n e^{-\beta_n} = \infty$ is stronger: it implies that $\beta_n = o(\log n)$ and is implied by $V_n(\varphi) = O(n^{-(1+\varepsilon)})$.

It is well known that the Variational Principle holds for the potential $\varphi = 0$; in fact

$$h_{top}(f) = P(0) = P_G(0) = P_{top}(0) = \lim_n \frac{1}{n} \log \operatorname{laps}(f^n),$$

where $\operatorname{laps}(f^n) := \#\mathcal{P}_n$ is the *lap number*, *i.e.,* the number of maximal intervals on which f^n is monotone, see [MSz]. In fact, the lap number is submultiplicative: $\operatorname{laps}(f^{n+m}) \leqslant \operatorname{laps}(f^n)\operatorname{laps}(f^m)$. Therefore $h_{top}(f) = \inf_n \frac{1}{n}\log\operatorname{laps}(f^n)$ and

$$e^{h_{top}(f)n} \leqslant \operatorname{laps}(f^n) \leqslant e^{n(h_{top}(f)+\varepsilon_n)}, \tag{29}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. We will extend this idea to ergodic averages of more general potentials in Lemma 11. For $J \in \mathcal{P}_m$, let $\varphi_m(J) = \sup\{\varphi_m(x) : x \in J\}$ and

$$Z_m^{top}(\varphi) := \sum_{J \in \mathcal{P}_m} e^{\varphi_m(J)}.$$

For the remainder of this section we assume that (I, f) is topologically mixing, *i.e.,* for each m, (I, f^m) is topologically transitive. In order to prove recurrence of φ, we need the following lemma.

Lemma 11. *Let φ be a potential satisfying* (1) *and with $\beta_n(\varphi) = o(n)$. Then there exists $\eta > 0$ such that $Z_n(\varphi) \geqslant \eta e^{-\beta_n} e^{P_{top}(\varphi)n}$ for all n, and $P_{top}(\varphi) = P_G(\varphi)$.*

Proof. Since f is topologically transitive, there is a collection of intervals permuted cyclically by f, such that for any interval J, there is n such that $f^n(J)$ contains a component of this cycle. For simplicity, let us assume that this collection is just a single interval I.

Since every m-cylinder set can contain at most one m-periodic point, $Z_m(\varphi) \leqslant Z_m^{top}(\varphi)$ for all m. Furthermore, $Z_m^{top}(\varphi)$ is submultiplicative, cf. (29), so

$$P_{top}(\varphi) := \lim_{m\to\infty} \frac{1}{m} \log Z_m^{top}(\varphi) = \inf_m \frac{1}{m} \log Z_m^{top}(\varphi) < \infty.$$

Therefore $P_G(\varphi) \leqslant P_{top}(\varphi) < \infty$.

Recall that every $J \in \mathcal{P}_m$ corresponds to a unique m-path $D_0 \to D_1 \to \cdots \to D_m$ in the Hofbauer tower $(\hat{I}, \hat{f})$ leading from the base D_0 of the tower to some terminal domain D_m. The level of D_m was defined as the length of the shortest path from the base to D_m. We say that the *pre-level* of J is pre-level$(J) = R$.

The topological entropy $h_{top}(f)$ is the exponential growth rate of the number of n-paths $D_0 \to \cdots \to D_n$ in the Hofbauer tower, and the limit of the exponential growth rates of the number of n-paths within $\hat{I}_R$ as $R \to \infty$, see [H2] for the unimodal and [BBr, Sections 9.3-9.4] for the general case. Therefore, by taking R sufficiently large, we can find $\gamma > 0$ and $C_0 \in (0, 1)$ such that the number of k-paths

$$\#\{D_0 \to D_1 \to \cdots \to D_k \ : \ \text{level}(D_k) \leqslant R, 1 \leqslant j \leqslant k\} \geqslant C_0 e^{k(h_{top}(f)-\gamma)} \tag{30}$$

for all $k \geqslant 1$, and

$$\sup \varphi - \inf \varphi < h_{top}(f) - \gamma - \frac{\log 2}{R}. \tag{31}$$

Since (I, f) is topologically transitive (and using our simplifying assumption), there exists R' depending on R, such that for each $D \in \mathcal{D}$ with level$(D) \leqslant R$, $f^{R'}(D) \supset I$. This implies that every $J \in \mathcal{P}_m$ with pre-level$(J) \leqslant R$. contains a periodic point of period $n := m + R'$.

The idea is now for an arbitrary $J \in \mathcal{P}_m$ to extend the corresponding path by R' arrows to find an n-periodic point $p \in J$. If pre-level$(J) \leqslant R$, then by the choice of R', this is indeed possible. We call such cylinder sets J *type 1*, and we can thus compare $Z_m^{\text{type }1}(\varphi)$ to $Z_n(\varphi)$ as:

$$\begin{aligned} Z_m^{\text{type }1}(\varphi) &= \sum_{J\in\mathcal{P}_m,\text{type }1} e^{\varphi_m(J)} \\ &\leqslant \sum_{\substack{p=f^n(p)\in J \\ J\in\mathcal{P}_m \text{ is type }1}} e^{\beta_m} e^{-R'\inf\varphi} e^{\varphi_n(p)} \leqslant e^{\beta_m} e^{-R'\inf\varphi} Z_n(\varphi). \end{aligned} \tag{32}$$

If pre-level$(J) > R$, then the existence of an n-periodic point in J cannot be guaranteed. We call such cylinder sets J *type 2*. Given such a type 2 cylinder set J, there is a maximal $m' < m$ such that pre-level$(J') = R$ for the m'-cylinder J' containing J. As we mentioned before, from any domain in the Hofbauer tower, there are at most two R-paths that are outside $\hat{I}_R$. Using this property repeatedly, we find that there are at most $2^{(m-m')/R} = e^{(m-m')\frac{\log 2}{R}}$ starting at $D_{m'}$ but otherwise outside $\hat{I}_R$. From $D_{m'}$, there is at least one R'-path leading back to some $D \in \hat{I}_R$, and using (31) and

(30) we derive that there are at least $C_0 e^{(m-m'-R')(h_{top}(f)-\gamma)}$ 'type 1' $m-m'$-paths from $D_{m'}$. From this we conclude that the type 1 cylinders "sufficiently" outnumber the type 2 cylinders, and we can bound the contributions of type 2 cylinders in J' by the contribution of type 1 cylinders in J' as follows:

$$\begin{aligned}\sum_{J\subset J',\text{type } 2} e^{\varphi_m(J)} &\leqslant e^{(m-m')(\sup\varphi+\frac{\log 2}{R})} \\ &\qquad \times \frac{1}{C_0}\, e^{-(m-m'-R')(h_{top}(f)-\gamma)} e^{-(m-m')\inf\varphi} \sum_{J\subset J',\text{type } 1} e^{\varphi_m(J)} \\ &\leqslant \frac{1}{C_0}\, e^{(m-m')(\sup\varphi-\inf\varphi-h_{top}(f)+\gamma+\frac{\log 2}{R})} e^{R'(h_{top}(f)-\gamma)} \sum_{J\subset J',\text{type } 1} e^{\varphi_m(J)} \\ &\leqslant \frac{1}{C_0}\, e^{R' h_{top}(f)} \sum_{J\subset J',\text{type } 1} e^{\varphi_m(J)}.\end{aligned}$$

Summing over all m' and $J' \in \mathcal{P}_{m'}$, we get

$$Z_m^{\text{type } 2}(\varphi) \leqslant \frac{1}{C_0}\, e^{R' h_{top}(f)} Z_m^{\text{type } 1}(\varphi).$$

Now we combine this with (32) and the fact that $\{Z_n^{top}(\varphi)\}_n$ is submultiplicative to obtain

$$\begin{aligned} e^{nP_{top}(\varphi)} &\leqslant Z_n^{top}(\varphi) \leqslant Z_{R'}^{top}(\varphi)\cdot Z_m^{top}(\varphi) \leqslant Z_{R'}^{top}(\varphi)\left[Z_m^{\text{type } 1}(\varphi) + Z_m^{\text{type } 2}(\varphi)\right] \\ &\leqslant Z_{R'}^{top}(\varphi)\left[1+\frac{1}{C_0}e^{R' h_{top}(f)}\right] Z_m^{\text{type } 1}(\varphi) \\ &\leqslant Z_{R'}^{top}(\varphi)\left[1+\frac{1}{C_0}e^{R' h_{top}(f)}\right] e^{-R'\inf\varphi} e^{\beta_m} Z_n(\varphi) \\ &\leqslant Z_{R'}^{top}(\varphi)\left(\frac{2}{C_0}\right) e^{R'(h_{top}(f)-\inf\varphi)} e^{\beta_m-\beta_n} e^{\beta_n} Z_n(\varphi) = \frac{1}{\eta} e^{\beta_n} Z_n(\varphi)\end{aligned}$$

for $\eta = \left(\frac{C_0}{2Z_{R'}^{top}(\varphi)}\right) e^{-R'(h_{top}(f)-\inf\varphi)} e^{\beta_n-\beta_m}$. Since $n-m=R'$, we can assume that $e^{\beta_m-\beta_n}$ is bounded independently of m, so $\eta > 0$. This proves the first statement. In fact, since $\beta_n = o(n)$, we also find $P_{top}(\varphi) = P_G(\varphi)$. $\square$

Corollary 2. *If $\sup\varphi - \inf\varphi < h_{top}(f)$ and $\sum_n e^{-\beta_n} = \infty$, then the potential φ is recurrent.*

Proof. Since φ is recurrent by definition if $\sum_n \lambda^{-n} Z_n(\varphi) = \infty$ for $\lambda = e^{P_G(\varphi)}$, this corollary is immediate from Lemma 11. $\square$

The above ideas lead us to show that in our setting P_{top} and P_G are in fact the same.

Corollary 3. *If $\sup\varphi - \inf\varphi < h_{top}(f)$, then $P_{top}(\hat\varphi) = P_G(\hat\varphi, \hat{\mathbf{C}})$ for every cylinder set $\hat{\mathbf{C}}$ in $\hat I_{trans}$.*

Proof. This is the same proof as Lemma 11 with $J \in \mathcal{P}_m$ replaced by $\hat J \in \hat{\mathcal{P}}_m \cap \hat{\mathbf{C}}$. $\square$

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