# DISTORTION BOUNDS FOR $C^{2+\eta}$ UNIMODAL MAPS 

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Abstract. We obtain estimates for derivative and cross-ratio distortion for $C^{2+\eta}$ (any $\eta>0$ ) unimodal maps with non-flat critical points. We do not require any 'Schwarzian-like' condition.

For two intervals $J \subset T$, the cross-ratio is defined as the value

$$
B(T, J):=\frac{|T||J|}{|L||R|}
$$

where $L, R$ are the left and right connected components of $T \backslash J$ respectively. For an interval map $g$ such that $g_{T}: T \rightarrow \mathbb{R}$ is a diffeomorphism, we consider the cross-ratio distortion to be

$$
B(g, T, J):=\frac{B(g(T), g(J))}{B(T, J)}
$$

We prove that for all $0<K<1$ there exists some interval $I_{0}$ around the critical point such that for any intervals $J \subset T$, if $\left.f^{n}\right|_{T}$ is a diffeomorphism and $f^{n}(T) \subset I_{0}$ then

$$
B\left(f^{n}, T, J\right)>K
$$

Then the distortion of derivatives of $\left.f^{n}\right|_{J}$ can be estimated with the Koebe Lemma in terms of $K$ and $B\left(f^{n}(T), f^{n}(J)\right)$. This tool is commonly used to study topological, geometric and ergodic properties of $f$. This extends a result of Kozlovski.

## 1. Introduction

In order to understand the long term behaviour of smooth dynamical system $f: X \rightarrow X$ we must consider iterates of the map. It is useful to know how differently high iterates of the map $f^{n}$ act on nearby points. For example we can try to estimate how wild the derivative of iterates of the map is: we can consider the distortion $\frac{D f^{n}(x)}{D f^{n}(y)}$ for $x, y$ in some small interval $J$ where $\left.f^{n}\right|_{J}$ is a diffeomorphism. For one dimensional maps, the Koebe Lemma is a tool we use to estimate this. Notice that this distortion can be rather wild when $f$ has critical points.

An important condition we must assume in order to apply the Koebe Lemma is that the map $f^{n}$ must increase cross-ratios. The type of cross-ratio we use most is defined as follows. For two intervals $J \subset T$, the cross-ratio is

[^0]defined as the value
$$
B(T, J):=\frac{|T||J|}{|L||R|}
$$
where $L, R$ are the left and right connected components of $T \backslash J$ respectively. For an interval map $g$ such that $g_{T}: T \rightarrow \mathbb{R}$ is a diffeomorphism, the main measure of cross-ratio distortion we use is given by
$$
B(g, T, J):=\frac{B(g(T), g(J))}{B(T, J)}
$$

If we know that $B\left(f^{n}, T^{*}, J^{*}\right) \geqslant K>0$ for any $J^{*} \subset T^{*} \subset T$ then we have uniform bounds on $\frac{D f^{n}(x)}{D f^{n}(y)}$ for $x, y \in J$ depending on $K$ and $B\left(f^{n}(T), f^{n}(J)\right)$. So we are able to estimate the distortion of the derivative of $f^{n}$ using information on the distortion of the cross-ratios.

A classical way of gaining information about the dynamics of an interval map $f:[0,1] \rightarrow[0,1]$ with a critical point, is to take a first return map to some well chosen interval $I$. If this map has some diffeomorphic branches, we can estimate how well or how badly the derivatives behave on branches using the Koebe Lemma as above. This method is often used to give information on the geometry and topology of the map and its iterates, see [MS]. This type of approach is also applied when considering the ergodic properties of one dimensional maps. Often instead of first return maps, certain inducing schemes are applied in these cases, see [MS]. The Koebe Lemma allows us to show that the inducing schemes are expansive, and the Folklore Theorem can then be used to derive ergodic absolutely continuous $f$-invariant measures.

In order to apply the Koebe Lemma to $\left.f^{n}\right|_{T}$ we need a lower bound on cross ratio distortion of $\left.f^{n}\right|_{T}$. In fact, a lower bound $K=1$ is obtained whenever $f$ is $C^{3}$ and has negative Schwarzian derivative: that is

$$
S f:=\frac{D^{3} f}{D f}-\frac{3}{2}\left(\frac{D^{2} f}{D f}\right)^{2}
$$

is negative wherever it is well defined. For applications it is not so important that $f$ have negative Schwarzian, just that some iterate of $f$ has negative Schwarzian on some small intervals. Kozlovski showed [K2] that for any $C^{3}$ unimodal map with non-flat critical point (see the next section), if $I$ is a small enough neighbourhood of the critical point and $f^{n}(x) \in I$ then $S f^{n+1}(x)<0$. Therefore, for most practical purposes, for example where first return maps or inducing schemes are used to gain information about the dynamics, it is unnecessary to find the sign of the Schwarzian derivative as long as the critical point is non-flat. Moreover, this result allowed Kozlovski to prove the following.

Theorem 1.1. Suppose that $f$ is a $C^{3}$ unimodal map with non-flat critical point whose iterates do not converge to a periodic attractor. Then for any
$0<K<1$, there is an interval $V$ around the critical point such that if, for an interval $T$ and some $n>0$,

- $\left.f^{n}\right|_{T}$ is monotone; and
- each interval from the orbit $\left\{T, f(T), \ldots, f^{n}(T)\right\}$ is contained in the domain of the first entry map to $V$,
then

$$
B\left(f^{n}, T, J\right)>K
$$

where $J$ is any subinterval of $T$.
This means that the Koebe Lemma can be applied to $f^{n}$ to get estimates on the distortion of derivatives which only depend on $B\left(f^{n}(T), f^{n}(J)\right)$ (for first return maps or induced maps this quantity is bounded whenever the branches have a 'uniform extension'). These results were extended to $C^{3}$ multimodal maps with non-flat critical points in $[\mathbf{S V}]$. Also, for $C^{3}$ unimodal maps with non-flat critical point, it is shown in [GSS2] that an analytic coordinate change can create a map which has first return maps with negative Schwarzian.

So how necessary is the negative Schwarzian condition to prove dynamical results in 'reasonable' cases? Certainly it is useful in determining the type of parabolic periodic points or bounding the number of attracting cycles, see [ $\mathbf{S i}, \mathbf{M S}]$. A natural question to ask, and the one we consider in this paper, is: what happens for unimodal maps with non-flat critical points which are not $C^{3}$ ? Certainly the usual negative Schwarzian condition is no use since it is not even defined. (Note that there is a 'Schwarzian-like' condition for $C^{1}$ maps -equivalent to the negative Schwarzian condition when the map is $C^{3}$ - see $[\mathbf{P}, \mathbf{M S}]$, but that need not hold in our case either.) We show that Theorem 1.1 extends to the case of $C^{2+\eta}$ for any $\eta>0$. So many results on the geometric and statistical properties of unimodal maps with non-flat critical point extend to maps which are only $C^{2+\eta}$.

Since we cannot use the negative Schwarzian property at all here, we must look rather closely at the behaviour of the map on small scales. We use a result in $[\mathbf{M S}]$ to estimate the cross ratio distortion in terms of sums of lengths of intervals. We split up this sum into blocks using the domains of first return maps to small intervals around the critical point. The precise behaviour of the branch containing the critical point, the central branch, determines how we choose our blocks. Since we have no negative Schwarzian property, there are particular difficulties when a block of our sum contains points which spend a very long time in the central branch (when there is a so called 'saddle node cascade' or an 'Ulam-Neumann cascade'). The main tool we use here is the real bounds proved by $[\mathbf{V}, \mathbf{S h 1}, \mathbf{S V}]$. Roughly speaking, these results give us a sequence of first return maps where the diffeomorphic branches have a uniformly large extension. This gives bounded distortion
of the derivative on these branches which allows us to estimate the sums of lengths of intervals.
1.1. Statement of the main result. We explain the terminology in the following definitions. Given an interval $T$, and a subinterval $J \subset T$, we defined the cross-ratio $B(T, J)$ above. Note that if we again denote the left-hand and right-hand components of $T \backslash J$ by $L$ and $R$ respectively, we have another measure of cross-ratio

$$
A(T, J):=\frac{|T||J|}{|L \cup J||J \cup R|},
$$

(however, we focus mainly on $B(T, J)$ ).
Suppose that $g: T \rightarrow \mathbb{R}$ is a diffeomorphism. We define $B(g, T, J)$ as above, but we also have

$$
A(g, T, J):=\frac{A(g(T), g(J))}{A(T, J)}
$$

another estimate of how the map distorts cross-ratios. Observe that for diffeomorphisms $g: T \rightarrow g(T)$ and $h: g(T) \rightarrow h \circ g(T)$ we have

$$
B(h \circ g, T, J)=B(h, g(T), g(J)) B(g, T, J)
$$

Similarly for $A(g, T, J)$.
We say that $T$ is a $\delta$-scaled neighbourhood of $J$ if $\frac{|L|}{|J|}, \frac{|R|}{|J|}>\delta$. We suppose throughout that our functions map from $I:=[0,1]$ into itself, and $\partial I$ into $\partial I$.

We say that a unimodal $C^{k}$ map $g$ has non-flat critical point $c$ if there exists some neighbourhood $U$ of $c$ and a $C^{k}$ diffeomorphism $\phi: U \rightarrow I$ with $\phi(c)=0$ such that $g(x)= \pm|\phi(x)|^{\alpha}+g(c)$ for some $\alpha>1$. The value $\alpha$ is known as the critical order for $g$. We denote the set of such maps by $N F^{k}$ and this neighbourhood by $U_{\phi}$.

Such maps have many good properties. For example, they have no wandering intervals, see for example Chapter IV of [MS]. More importantly for us here is how such maps distort cross-ratios. In particular, how iterates of such maps distort cross-ratios. Our main result is as follows.
Theorem 1.2. For any $\eta>0$, let $f \in N F^{2+\eta}$ be a unimodal map with a critical point whose iterates do not converge to a periodic attractor. Then for any $0<K<1$, there is an interval $V$ around the critical point such that if, for an interval $T$ and some $n>0$,

- $\left.f^{n}\right|_{T}$ is monotone; and
- $f^{n}(T) \subset V$,
then

$$
B\left(f^{n}, T, J\right)>K
$$

$$
A\left(f^{n}, T, J\right)>K
$$

where $J$ is any subinterval of $T$.
This theorem is proved for $C^{3}$ maps in [K2]. Note that in fact we prove that if $0<\eta \leqslant 1$ then for any $0<\eta^{\prime}<\eta$, there exists $C>0$ such that if $J, T, V$ are as in the theorem then $A\left(f^{n}, T, J\right), B\left(f^{n}, T, J\right)>\exp \left\{-C\left(\sup _{j}\left|V_{j}\right|\right)^{\eta^{\prime}}\right\}$.
1.2. Strategy of the proof. Our setup will involve first return maps to a neighbourhood of $c$, as outlined below. For the case where $c$ is non-recurrent see $[\mathbf{S t}]$. So we suppose throughout that $c$ is recurrent.

An open interval $V$ is nice for $f$ if $f^{n}(\partial V) \cap V=\emptyset$ for $n \geqslant 1$. (When it is clear what $f$ is, we just refer to such interval as nice.) It is easy to see that we can find arbitrarily small nice intervals around $c$.

Let $I_{0} \ni c$ be a nice interval. For every $x \in I$ whose orbit intersects $I_{0}$, let $n(x):=\min \left\{k>0: f^{k}(x) \in I_{0}\right\}$. If additionally $x \in I_{0}$, let $I_{0}^{j} \ni x$ be the maximal neighbourhood such that $f^{n(x)}\left(I_{0}^{j}\right) \subset I_{0}$. We obtain the first return map $F_{0}: \bigcup_{j} I_{0}^{j} \rightarrow I_{0}$. We label the interval which contains $c$ by $I_{0}^{0}$; this interval is called the central domain. Observe that $F_{0}$ is a diffeomorphism on all domains $I_{0}^{j}$ except when $j=0 . F_{0}$ is unimodal on $I_{0}^{0}$. Note also that $I_{0}^{0}$ is again a nice interval. We will call it $I_{1}$ for the next step in the inducing process; i.e. we define $F_{1}: \bigcup_{j} I_{1}^{j} \rightarrow I_{1}$ to be the first return map to $I_{1}=I_{0}^{0}$. It has central domain $I_{1}^{0}=I_{2}$. Continuing inductively, we obtain maps $F_{i}: \bigcup_{j} I_{i}^{j} \rightarrow I_{i}$. The sequence $I_{0} \supset I_{1} \supset \cdots$ is called the principal nest, and $\left.F_{i}\right|_{I_{i}^{j}}: I_{i}^{j} \rightarrow I_{i}$ is a branch of $F_{i}$.

If $x \notin I_{i}$ but $n(x)$ is defined then there is a maximal interval $U_{i}^{j} \ni x$ such that $f^{n(x)}: U_{i}^{j} \rightarrow I_{i}$ is a diffeomorphism. So we may extend $F_{i}$, letting $\left.F_{i}\right|_{U_{i}^{j}}: U_{i}^{j} \rightarrow I_{i}$. Then letting $\cup_{j} U_{i}^{j}$ consist of all such intervals added to $\bigcup_{j} I_{i}^{j}$, we call $F_{i}: \bigcup_{j} U_{i}^{j} \rightarrow I_{i}$ the first entry map to $I_{i}$. We will often switch between these two very similar types of map.

For simplicity, except in the appendix, we will assume that $F_{i}(c)$ is a maximum for $\left.F_{i}\right|_{I_{i+1}}$. We say that $F_{i}$ is low if $F_{i}(c)$ lies to the left of $c$ and $F_{i}$ is high if $F_{i}(c)$ lies to the right of $c . F_{i}$ is central if $F_{i}(c)$ is inside $I_{i+1}$ (if this is not the case, then $F_{i}$ is non-central). Figure 1 shows $F_{i}$ which is high and central return.

Suppose that $f^{n}: T \rightarrow f^{n}(T)$ is a diffeomorphism and $f^{n}(T) \subset I_{0}$. It can be shown (see Lemma 2.1) that we get a lower bound on $B\left(f^{n}, T, J\right)$ if we can find some bound on $\sum_{k=0}^{n-1}\left|f^{k}(T)\right|$. In fact, we consider $\sum_{k=0}^{n-1}\left|f^{k}(T)\right|^{1+\xi}$ for some $0<\xi<\eta$. We will split up this sum into blocks determined by the principal nest explained above. Note that our proofs extend easily to $A\left(f^{n}, T, J\right)$, see $[\mathbf{S t}]$.


Figure 1. $F_{i}$ is high and central.

We fix $n$ and $T$ as in Theorem 1.2, let $n_{0}=n$. For $i>0$, suppose that some iterate $f^{j}(T)$ enters $I_{i}$ for $0 \leqslant j \leqslant n$. Now we let $n_{i}$ be the last time that $f^{j}(T) \subset I_{i}$, i.e. $f^{n_{i}}(T) \subset I_{i}$ and $f^{n_{i}+j}(T) \nsubseteq I_{i}, 0<j \leqslant n-n_{i}$. If $f^{j}(T)$ is never contained in $I_{i}$ for $0 \leqslant j \leqslant n$ then we let $n_{i}=n_{i-1}$. For each $i$, we will be interested in estimating

$$
\sum_{k=1}^{n_{i}-n_{i+1}}\left|f^{k+n_{i+1}}(T)\right|^{1+\xi} \quad \text { we call this the the sum for } F_{i} .
$$

As we will see later, if $F_{i}$ is non-central infinitely often then Theorem 2.3 implies that as $i \rightarrow \infty$ the intervals $I_{i}$ shrink down to $c$. Thus we are able to bound $\sum_{k=0}^{n-1}\left|f^{k}(T)\right|^{1+\xi}$ by bounding the sums for all $F_{i}$. We will use a slightly different method when there exists a nice $I_{0}$ such that $F_{i}$ is always central.

In order to prove the main theorem, we will consider the following cases. Note that we only assume that $f \in N F^{2}$ in the following three propositions.

- $F_{i-2}$ is non-central. We consider the sum for $F_{i}$ whenever $f^{j}(T) \cap$ $\partial I_{i+1}=\emptyset$ for all $0 \leqslant j<n_{i}$, as follows.

Proposition 1.3. Suppose that $F_{i-2}$ is non-central and $f^{j}(T) \cap$ $\partial I_{i+1}=\emptyset$ for all $0 \leqslant j<n_{i}$. Then there exists $C_{w b}>0$ such that

$$
\sum_{k=1}^{n_{i}-n_{i+1}}\left|f^{k+n_{i+1}}(T)\right|<C_{w b} \sigma_{i} \frac{\left|f^{n_{i}}(T)\right|}{\left|I_{i}\right|}
$$

where $\sigma_{i}:=\sup _{V \in\left\{I_{i}^{j}\right\}_{j}} \sum_{k=1}^{n(V)}\left|f^{k}(V)\right|$ (and $n(V)$ is defined as $k$ where $\left.\left.F_{i}\right|_{V}=f^{k}\right)$.

We call this a well bounded case. It is dealt with in Section 3.

- $F_{i-2}$ is non-central and $F_{i}, \ldots, F_{i+m-1}$ are central. We consider the sums for $F_{i}, F_{i+1}, \ldots, F_{i+m}$ whenever $f^{j}(T) \cap \partial I_{i+m+1}=\emptyset$ for all $0 \leqslant j<n_{i}$, as follows.

Proposition 1.4. Suppose that $F_{i-2}$ is non-central, $F_{i}, \ldots, F_{i+m-1}$ are central and $f^{j}(T) \cap \partial I_{i+m+1}=\emptyset$ for all $0 \leqslant j<n_{i}$. For all $\xi>0$ there exists $C_{\text {casc }}>0$ such that

$$
\sum_{k=1}^{n_{i}-n_{i+m+1}}\left|f^{k+n_{i+m+1}}(T)\right|^{1+\xi}<C_{c a s c} \sigma_{i, m} \max _{n_{i+m+1}<k \leqslant n_{i}}\left|f^{k}(T)\right|^{\xi}
$$

where $\sigma_{i, m}$ is defined as follows. Let $\sigma_{i}:=\sup _{V \in\left\{I_{i}^{j}\right\}_{j}} \sum_{k=1}^{n(V)}\left|f^{k}(V)\right|$. Let $\hat{V} \subset I_{i} \backslash I_{i+1}$ be an interval such that $f^{\hat{n}}(\hat{V})$ is one of the connected components of $I_{i} \backslash I_{i+1}$ for some $\hat{n}>0$ and $f^{j}(\hat{V})$ is disjoint from both $I_{i} \backslash I_{i+1}$ and $I_{m}$ for $0<j<\hat{n}(\hat{V})$. Then $\sigma_{i, m}$ is the supremum of all such sums $\sum_{j=1}^{\hat{n}(\hat{V})}\left|f^{j}(\hat{V})\right|$ and $\sigma_{i}$.

We call this the cascade case. It is dealt with in Section 4.

- $F_{i-2}$ is central and $F_{i-1}$ is high and non-central. We consider the sum for $F_{i}$ whenever $f^{j}(T) \cap \partial I_{i+1}=\emptyset$ for all $0 \leqslant j<n_{i}$, as follows.
Proposition 1.5. Suppose that $F_{i-2}$ is central, $F_{i-1}$ is high and non-central and $f^{j}(T) \cap \partial I_{i+m+1}=\emptyset$ for all $0 \leqslant j<n_{i}$. Then there exist $C_{e x}>0$ and $n_{i+1}<n_{i, 3}<n_{i, 2}<n_{i}$ such that $f^{n_{i, 2}}(T), f^{n_{i, 3}}(T) \subset$ $I_{i}$ and

$$
\sum_{k=1}^{n_{i}-n_{i+1}}\left|f^{k+n_{i+1}}(T)\right|<C_{e x} \sigma_{i}\left(\frac{\left|f^{n_{i}}(T)\right|}{\left|I_{i}\right|}+\frac{\left|f^{n_{i, 2}}(T)\right|}{\left|I_{i}\right|}+\frac{\left|f^{n_{i, 3}}(T)\right|}{\left|I_{i}\right|}\right) .
$$

(In some cases, the last two terms in the sum are not required.) We call this the exceptional branches case. It is dealt with in Section 5 . We also note there that if $F_{i-2}$ is central and $F_{i-1}$ is low and non-central then we are in another well bounded case, and so the conclusion of Proposition 1.3 holds.

- We have an interval $I_{0}$ such that $F_{i}$ are all central for $i=0,1, \ldots$. We call this the infinite cascade case. We prove Theorem 1.2 for this case in Section 7.

The proof of Theorem 1.2 for the non-infinite cascade case is given in Section 6.

With these propositions, for $0<\eta^{\prime}<\eta$, we can decompose the sum $\sum_{k=0}^{n-1}\left|f^{k}(T)\right|^{1+\eta^{\prime}}$ into blocks of sums $\sum_{k=1}^{n_{i}-n_{i+1}}\left|f^{k+n_{i+1}}(T)\right|^{1+\eta^{\prime}}$. We then show that each of these is uniformly bounded. We will then show that $\sum_{k=1}^{n_{i}-n_{i+1}}\left|f^{k+n_{i+1}}(T)\right|^{1+\eta}$ decays in a uniform way with $i$.

The first two cases use real bounds of Theorem 2.3. These bounds imply that $B\left(I_{i}^{j}, I_{i}\right)$ are bounded above. This will also be true for all except possibly
two domains of $F_{i}$ in the third case. The main tool here is Lemma 3.3, which gives us some decay of cross-ratios when we have these real bounds. Note that the conditions $f^{j}(T) \cap \partial I_{i+1}=\emptyset$ for all $0 \leqslant j<n_{i}$ in well bounded and exceptional cases, and $f^{j}(T) \cap \partial I_{i+m+1}=\emptyset$ for all $0 \leqslant j<n_{i}$ in the cascade case, make the propositions simpler to prove. However, as we remark in Section 6, it is easy to see how to split up the intervals in the other cases in order to prove Theorem 1.2.

The final case, which arises in the infinitely renormalisable case, is different from the other three. We use a lemma of [K2] to find some uniform expanding property which helps bound the sums.

In all cases except the infinite cascade case we must ensure that we have some initial interval which has a first return map which is well bounded. To do this we can simply pick some nice interval to begin with and then induce until we find a map which is well-bounded. This is always possible when there is not an infinite cascade.

Note that we need extra smoothness to bound cross-ratios in the cascade case. This ensures that we can deal with the case when we have many consecutive low central returns, a 'saddle node cascade'.

In Kozlovski's proof for $C^{3}$ maps he was able to use the fact that there exists some $C>0$ depending only on $f$ such that for interval $J \subset T$ we have $B(f, T, J)>\exp \left\{-C|T|^{2}\right\}$ and $A(f, T, J)>\exp \{C|L||R|\}$. See Chapter IV. 2 of [MS]. In particular this means that there exist such real bounds as in Theorem 2.3 for all $i$, not just those for which $F_{i-1}$ is a non-central return. So the long central cascades we encounter in Section 4 present much less of a problem in the $C^{3}$ case. Indeed, the work done in Section 5 is also unnecessary in the $C^{3}$ case.

We will deal with the well bounded case first. It is the simplest and gives us a good idea about how we may proceed in general. We will use $J$ to refer to a general interval from here until Section 6. This allows us to use less notation. When we use the constant $C>0$, we mean some constant depending only on $f$.

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## 2. Introductory results

Without loss of generality, we suppose throughout that our maps have a maximum at the critical point. We also suppose that $f$ is symmetric about $c$. That is, $f(c-\epsilon)=f(c+\epsilon)$ for all $\epsilon$. This assumption is useful for simplifying proofs (particularly in Section 5, which is already quite technical), but is not crucial since on small scales our maps will be essentially symmetric (in particular, $|D f(c-\epsilon)|$ and $|D f(c+\epsilon)|$ are arbitrarily close for small enough $\epsilon$ ). We let $C \leqslant|g|_{U} \leqslant C^{\prime}$ mean $\sup _{x \in U}|g(x)| \leqslant C^{\prime}$ and $\inf _{x \in U}|g(x)| \geqslant C$.

The following theorem is proved for a more general case in Chapter IV of [MS]. Here we will let $w_{g}$ be the modulus of continuity of a continuous map $g$, i.e. $w_{g}(\epsilon):=\sup _{|x-y|<\epsilon}|g(x)-g(y)|$.
Theorem 2.1. For a unimodal map $g: I \rightarrow I, g \in N F^{2}$, if $T$ is an interval such that $\left.g^{n}\right|_{T}$ is a diffeomorphism and $J \subset T$ is a subinterval, then there exists some $C>0$ such that

$$
B\left(g^{n}, T, J\right)>\exp \left\{-C \sum_{i=0}^{n-1} w_{D^{2} g}\left(\left|g^{i}(T)\right|\right)\left|g^{i}(T)\right|\right\}
$$

This bound also holds for $A\left(f^{n}, T, J\right)$.
In Sections 6 and 7 we will use the fact that when $g \in N F^{2+\eta}$ for some $\eta>0$, we can replace $C w_{D^{2} g}(\epsilon)$ by $C \epsilon^{\eta}$.

The following lemma, a consequence of the absence of wandering intervals, is Lemma 5.2 in [K2].
Lemma 2.2. Suppose that $g \in N F^{2}, g: I \rightarrow I$. Then there exists a function $\tau:[0,|I|] \rightarrow[0, \infty)$ such that $\lim _{\epsilon \rightarrow 0} \tau(\epsilon)=0$ and for any interval $V$ for which $\left.g^{n}\right|_{V}$ is a diffeomorphism and $g^{n}(V)$ is disjoint from the immediate basins of periodic attractors, we have

$$
\max _{0 \leqslant i \leqslant n}\left|g^{i}(V)\right|<\tau\left(\left|g^{n}(V)\right|\right)
$$

We may use this lemma and Theorem 2.1 to get

$$
\begin{equation*}
B\left(g^{n}, T, J\right)>\exp \left\{-\sigma^{\prime}\left(\left|g^{n-1}(T)\right|\right) \sum_{i=0}^{n-1}\left|g^{i}(T)\right|\right\} \tag{1}
\end{equation*}
$$

whenever $f^{n}(T)$ is disjoint from the immediate basins of periodic attractors, where

$$
\begin{equation*}
\sigma^{\prime}\left(\left|g^{m}(T)\right|\right)=C w_{g} \circ \tau\left(\left|g^{m}(T)\right|\right) \tag{2}
\end{equation*}
$$

We will use the following result of $[\mathbf{S V}]$ throughout. (In fact it is stated there in greater generality, as Theorem A.)

Theorem 2.3. If $g \in N F^{2}$ is a unimodal map with recurrent critical point, then the following hold.
(a) For all $k \geqslant 0$ there exists $\xi(k)>0$ such that if $G_{i-1}: \bigcup_{j} I_{i-1}^{j} \rightarrow I_{i-1}$ is non-central, then $I_{i+k}$ is a $\xi(k)$-scaled neighbourhood of $I_{i+k+1}$.
(b) For each $\xi>0$ there is some $\xi>0$ such that if $I_{i}$ is a $\xi$-scaled neighbourhood of $I_{i+1}$ then $I_{i+1}$ is a $\hat{\xi}$-scaled neighbourhood of any domain of $G_{i+1}$.

This result gives us real bounds for some of our first return maps. We let $\chi:=\xi(1)>0$ from the above theorem for our map $f$.

The following theorem is an improvement of the classical Koebe Lemma. It is presented in more generality in $[\mathbf{S V}]$ as Proposition 2: 'a Koebe principle requiring less disjointness'. Note that actually for our purposes, the classical Koebe Lemma is enough.

Theorem 2.4. Suppose that $g \in N F^{2}$. Then there exists a function $\nu$ : $[0,|I|] \rightarrow[0, \infty)$ such that $\nu(\epsilon) \rightarrow 0$ as $\epsilon \rightarrow 0$ with the following properties. Suppose that for some intervals $J \subset T$ and a positive integer $n$ we know that $\left.g^{n}\right|_{T}$ is a diffeomorphism. Suppose further that $g^{n}(T)$ is a $\delta$-scaled neighbourhood of $g^{n}(J)$ for some $\delta>0$. Then,
(a) for every $x, y \in J$,

$$
\frac{\left|D g^{n}(x)\right|}{\left|D g^{n}(y)\right|}<\exp \left\{\nu(S(n, T)) \sum_{i=0}^{n-1}\left|g^{i}(J)\right|\right\}\left[\frac{1+\delta}{\delta}\right]^{2}=: C(\delta)
$$

where $S(n, T):=\max _{0 \leqslant k \leqslant n-1}\left|f^{k}(T)\right|$.
(b) $T$ is a $\tilde{\delta}$-scaled neighbourhood of $J$ whenever

$$
\tilde{\delta}:=\frac{1}{2} \exp \{-\theta\}\left[\frac{1+\delta}{\delta}\right]^{2}\left(\frac{-2 \theta+\delta(1-2 \theta)}{2+\delta}\right)
$$

is positive, where $\theta:=\nu(S(n, T)) \sum_{i=0}^{n-1}\left|g^{i}(J)\right|$.

Again we may use Lemma 2.2 to substitute $\nu(S(n, T))$ with $\nu^{\prime}\left(\left|f^{n}(T)\right|\right)$ where we define $\nu^{\prime}\left(\left|f^{m}(V)\right|\right):=\nu \circ \tau\left(\left|f^{m}(V)\right|\right)$. We will use the result of Theorem 2.3(b) extensively, but we use $\tilde{\delta}$ when $\theta=\nu^{\prime}\left(\left|I_{0}\right|\right)$. Usually $\delta$ will be related to the $\chi$ we obtained following Theorem 2.3.

We will sometimes be in a situation where we wish to estimate the derivative of a function in between two points at which we know something about the derivative. The following two well known results allow us to do this. The following is known as the Minimum Principle; see, for example, Theorem IV.1.1 of [MS].

Theorem 2.5. Let $T=[a, b] \subset I$ and $g: T \rightarrow g(T) \subset I$ be a $C^{1}$ diffeomorphism. Let $x \in(a, b)$. If for any $J^{*} \subset T^{*} \subset T$,

$$
B\left(g, T^{*}, J^{*}\right)>\mu_{g}>0
$$

then

$$
|D g(x)|>\mu_{g}^{3} \min (|D g(a)|,|D g(b)|)
$$

To see a proof of the following well known result see [MS].
Theorem 2.6. For $g \in N F^{2}$ there exist $n_{0} \in \mathbb{N}$ and $\rho_{g}>1$ such that if $p$ is a periodic point of period $n \geqslant n_{0}$ then $\left|D g^{n}(p)\right|>\rho_{g}$.

We are now ready to begin the proof of Theorem 1.2.

## 3. Well bounded case

Here we deal with the case where $F_{i-2}$ is non-central and $f^{j}(T) \cap \partial I_{i+1}=\emptyset$ for all $0 \leqslant j<n_{i}$. In our estimates, we are principally interested in iterates of $T$ landing in $I_{i}^{j}$ for $j \neq 0$. By Theorem 2.3, the fact that $F_{i-2}$ is noncentral implies that the first return domains $I_{i}^{j}$ are all well inside $I_{i}$. This enables us to estimate the sum for $F_{i}$, and is the reason we call this case well bounded.

Let $n_{i}^{\prime}>n_{i+1}$ be minimal such that $f^{n_{i}^{\prime}}(T) \subset I_{i}$. We will initially assume that we have some $\kappa>0$ such that for the 'return sum',

$$
\begin{equation*}
\sum_{k=0}^{j_{i}}\left|F_{i}^{k}\left(f^{n_{i}^{\prime}}(T)\right)\right|<\kappa\left|f^{n_{i}}(T)\right| \tag{3}
\end{equation*}
$$

where $j_{i}$ is such that $\left.F^{j_{i}}\right|_{f^{n_{i}^{\prime}}(T)}=\left.f^{n_{i}-n_{i}^{\prime}}\right|_{f^{n_{i}^{\prime}(T)}}$. We prove Proposition 1.3 before bounding this return sum in order to give an idea why we need bounds on return sums. Except for the proof of (3), this is similar to the proof of Lemma 5.3.4 of $[\mathbf{K 1}]$. There, it is assumed that $f \in C^{3}$ in order to bound the sum $\sum_{k=0}^{j_{i}-1}\left|F_{i}^{k}\left(f^{n_{i}^{\prime}}(T)\right)\right|$. Those methods fail in the $C^{2}$ case.

Proof of Proposition 1.3 assuming (3). Let $n_{i+1}=m_{0}<m_{1}<\cdots<$ $m_{j_{i}}=n_{i}$ be all the integers between $n_{i+1}$ and $n_{i}$ such that $f^{m_{j}}(T) \subset I_{i} \backslash I_{i+1}$ for $j=1, \ldots, j_{i}-1$ and let $m_{0}=n_{i+1}$. Now let $F_{i}: \bigcup_{j} U_{i}^{j} \rightarrow I_{i}$ be the first entry map to $I_{i}$. We will decompose $\sum_{k=1}^{n_{i}-n_{i+1}}\left|f^{k+n_{i+1}}(T)\right|$ as $\sum_{j=0}^{j_{i}-1} \sum_{k=1}^{m_{j+1}-m_{j}}\left|f^{k+m_{j}}(T)\right|$.

For $1 \leqslant j \leqslant j_{i}-1$ and $1 \leqslant k<m_{j+1}-m_{j}$, let $U_{i}^{l}$ be the domain of first entry to $I_{i}$ such that $f^{m_{j}+k}(T) \subset U_{i}^{l}$. Suppose that $\left.F_{i}\right|_{U_{i}^{l}}=f^{i_{l}}$. Then there exists an extension to $V_{i}^{l} \supset U_{i}^{l}$ so that $f^{i_{l}}: V_{i}^{l} \rightarrow I_{i-1}$ is a diffeomorphism.

Then by the Koebe Lemma we have the distortion bound: $\frac{\left.\mid f^{k+m_{j}}(T)\right) \mid}{\left|U_{i}^{l}\right|} \leqslant$ $C(\chi) \frac{\left|f^{m_{j+1}}(T)\right|}{\left|I_{i}\right|}$. Whence

$$
\begin{aligned}
\sum_{k=1}^{m_{j+1}-m_{j}}\left|f^{m_{j}+k}(T)\right| & \leqslant C(\chi)\left(\frac{\left|f^{m_{j+1}}(T)\right|}{\left|I_{i}\right|}\right) \sum_{k=0}^{m_{j+1}-m_{j}-1}\left|f^{k}\left(U_{i}^{j}\right)\right| \\
& \leqslant C(\chi) \sigma_{i} \frac{\left|f^{m_{j+1}}(T)\right|}{\left|I_{i}\right|} .
\end{aligned}
$$

Therefore

$$
\sum_{k=1}^{n_{i}-n_{i+1}}\left|f^{k+n_{i+1}}(T)\right| \leqslant C(\chi) \frac{\sigma_{i}}{\left|I_{i}\right|} \sum_{j=1}^{j_{i}}\left|f^{m_{j}}(T)\right|=C(\chi) \frac{\sigma_{i}}{\left|I_{i}\right|} \sum_{k=0}^{j_{i}-1}\left|F_{i}^{k}(\hat{T})\right|
$$

where $\hat{T}:=f^{n_{i}^{\prime}}(T)$. This is bounded above by $\kappa\left|f^{n_{i}}(T)\right|$ due to (3), so we are finished.
3.1. Bounding return sums. In this subsection we will introduce some tools which we use extensively in the remainder of this paper. We then use these tools to prove that (3) holds.

The proof of the following simple lemma is left to the reader.
Lemma 3.1. For all $\delta>0$ there exists $\Delta=\Delta(\delta)>0$ such that $\Delta(\delta) \rightarrow 0$ as $\delta \rightarrow \infty$ with the following property. Suppose that $U$ is an interval, $J \subset U$ is a subinterval and that the left and right components of $U \backslash J$ are denoted by $L$ and $R$ respectively. Suppose further that $|L|,|R|>\delta|J|$. Then

$$
B(U, J)<\Delta
$$

Let $D_{1}$ denote the set of non-central domains $F_{i}^{-1}\left(I_{i}\right)$, i.e. $D_{1}=\bigcup_{j \neq 0} I_{i}^{j}$. Let $D_{2}$ denote the set of domains $F_{i}^{-1}\left(D_{1}\right)$ which are disjoint from the central domain. Inductively, we let $D_{k}$ denote the set of domains $F_{i}^{-1}\left(D_{k-1}\right)$ which are disjoint from the central domain. Then for any element $J_{k} \in D_{k}$, $F_{i}^{k}: J_{k} \rightarrow I_{i}$ is a diffeomorphism. We will bound $\sum_{j=0}^{k-1}\left|F_{i}^{j}\left(J_{k}\right)\right|$ for any $J_{k} \in D_{k}$ by showing that there exists some $\lambda<1$ independent of $i$ such that for $k>1$ we have $B\left(I_{i}, J_{k}\right) \leqslant \lambda B\left(I_{i}, F_{i}\left(J_{k}\right)\right)$. We let

$$
\begin{equation*}
\mu:=\exp \left\{-\sigma^{\prime}\left(\left|I_{0}\right|\right)\right\} \tag{4}
\end{equation*}
$$

where $\sigma^{\prime}$ is given by (2). By (1), if $J^{\prime}, f\left(J^{\prime}\right), \ldots, f^{m}\left(J^{\prime}\right)$ is a disjoint set of intervals and $J \supset J^{\prime}$, we have $B\left(f^{m}, J^{\prime}, J\right)>\mu$. Therefore, if $n(j)$ is the return time of $I_{i}^{j}$ to $I_{i}$ and $J \subset I_{i}^{j}$ then $B\left(f^{n(j)}, I_{i}^{j}, J\right)>\mu$.

The following lemma is Lemma 2.3 of [GK].

Lemma 3.2. For every $\delta>0$ there exists $\lambda^{\prime}=\lambda^{\prime}(\delta)<1$ such that if $J \subset V \subset U$ are intervals and $U$ is a $\delta$-scaled neighbourhood of $V$ then

$$
B(U, J)<\lambda^{\prime} B(V, J)
$$

Furthermore, $\lambda^{\prime} \rightarrow 1$ as $\delta \rightarrow 0$.

We add this lemma to (1) as follows.
Lemma 3.3. Given $\delta>0$, there exist $0<\lambda=\lambda(\delta)<1$ and $\epsilon>0$ such that if $\left|I_{0}\right|<\epsilon$ and $I_{i-1}$ is a $\delta$-scaled neighbourhood of $I_{i}$, then for any $J \subset I_{i}^{j}$ with $j \neq 0$,

$$
B\left(I_{i}, J\right)<\lambda B\left(I_{i}, F_{i}(J)\right)
$$

Proof. From the previous lemma there exists some $\lambda^{\prime}=\lambda^{\prime}(\delta)<1$ such that

$$
B\left(I_{i}, J\right)<\lambda^{\prime} B\left(I_{i}^{j}, J\right)
$$

Now from (1) we obtain

$$
B\left(I_{i}, J\right)<\lambda^{\prime} \frac{B\left(I_{i}, F_{i}(J)\right)}{\mu}
$$

where $\mu$ is defined in (4). Since $\mu \rightarrow 1$ as $\left|I_{0}\right| \rightarrow 0$, if $\epsilon$ is chosen small enough then $\frac{\lambda^{\prime}}{\mu}<1$. We let $\lambda:=\frac{\lambda^{\prime}}{\mu}$. Thus $B\left(I_{i}, J\right)<\lambda B\left(I_{i}, F_{i}(J)\right)$.

We will consider $\lambda=\lambda(\tilde{\chi})$ where $\tilde{\chi}$ comes from Theorem 2.4(b) applied to $\chi$ and $\chi$ comes from Theorem 2.3(a), i.e. $\tilde{\chi}$ takes the role of $\delta$ in Lemma 3.3. In fact we shall adjust $\lambda$ again in Section 5 , but it will remain independent of $i$ and strictly less than 1 .

Proof of (3). For $k \geqslant 2, B\left(I_{i}, J_{k}\right)<\lambda^{k-1} B\left(I_{i}, F_{i}^{k-1}\left(J_{k}\right)\right)$. Suppose that $F_{i}^{k-1}\left(J_{k}\right) \subset I_{i}^{j}$. Then by Lemma 3.1, using Theorems 2.3 and 2.4 (b), $B\left(I_{i}, I_{i}^{j}\right)<\Delta$ where $\Delta=\Delta(\tilde{\chi})$. Thus, it is easy to see $B\left(I_{i}, F_{i}^{k-1}\left(J_{k}\right)\right)<$ $\Delta \frac{\left|F_{i}^{k-1}\left(J_{k}\right)\right|}{\left|I_{i}^{j}\right|}$. Now by the Koebe Lemma, $\left|F_{i}^{k-1}\left(J_{k}\right)\right|<C(\chi)\left|F_{i}^{k}\left(J_{k}\right)\right| \frac{\left|I_{i}^{j}\right|}{\left|I_{i}\right|}$, so we know that $B\left(I_{i}, F_{i}^{k-1}\left(J_{k}\right)\right)<C(\chi) \Delta \frac{\left|F_{i}^{k}\left(J_{k}\right)\right|}{\left|I_{i}\right|}$. We apply these estimates to the sizes of $J_{k}$ :

$$
\left|J_{k}\right|<\frac{\left|I_{i}\right|}{1+\frac{2\left|I_{i}\right|}{\left.\lambda^{k-1} C(\chi) \Delta \mid F_{i}^{k( } J_{k}\right) \mid}}
$$

Then $\left|J_{k}\right|<C \lambda^{k-1}\left|F_{i}^{k}\left(J_{k}\right)\right|$. So $\sum_{j=0}^{k-1}\left|F_{i}^{j}\left(J_{k}\right)\right|<C \frac{\left|F_{i}^{k}\left(J_{k}\right)\right|}{1-\lambda}$. Whence

$$
\sum_{j=0}^{k}\left|F_{i}^{j}\left(J_{k}\right)\right|<\left|F_{i}^{k}\left(J_{k}\right)\right|\left(1+\frac{C}{1-\lambda}\right)
$$

This holds for any sum of returns which never lands in the central domain. It is independent of $i$. Letting $\kappa=\left(1+\frac{C}{1-\lambda}\right)$ we prove (3).

## 4. Cascade case

This section is devoted to the proof of Proposition 1.4. Note that if there is a uniform upper bound on the length of sequences $F_{i}, F_{i+1}, \ldots, F_{i+m}$ all having central returns then Theorem 2.3 implies that we may prove Proposition 1.4 as a well bounded case. However, there may be arbitrarily long sequences of consecutive central returns.

Proof of Proposition 1.4. We suppose that there $i$ is such that $f^{n_{i}}(T) \subset I_{i}$ where $F_{i-2}$ has a non-central return and $F_{i+j}$ all have central returns for $j=0, \ldots, m-1$ and that $F_{i+m}$ has a non-central return. For $\xi>0$ we will bound the sum

$$
\sum_{k=1}^{n_{i}-n_{i+m+1}}\left|f^{k+n_{i+m+1}}(T)\right|^{1+\xi} .
$$

For our intial estimates, we may omit $\xi$, but later it will be necessary to include it. Recall that we always assume here that $f^{j}(T) \cap \partial I_{i+m+1}=\emptyset$ for all $0 \leqslant j<n_{i}$.

Let $m_{0}=n_{i+m+1}$ and let $m_{0}<m_{1} \leqslant n_{i}$ be the smallest integer such that $f^{m_{1}}(T) \subset I_{i} \backslash I_{i+1}$. Let $m_{1}<m_{2} \leqslant n_{i}$ be the next integer for which $f^{m_{2}}(T) \subset I_{i} \backslash I_{i+1}$ if such $m_{2}$ exists. Proceeding in this manner, we obtain a sequence, $n_{i+m+1}<m_{1}<m_{2}<\cdots<m_{N}=n_{i}$.

So

$$
\sum_{k=1}^{n_{i}-n_{i+m+1}}\left|f^{k+n_{i+m+1}}(T)\right|=\sum_{j=0}^{N-1} \sum_{k=1}^{m_{j+1}-m_{j}}\left|f^{k+m_{j}}(T)\right| .
$$

Define $m_{N-1}<m^{\prime} \leqslant n_{i}$ to be minimal such that $f^{m^{\prime}}(T) \subset I_{i} \backslash I_{i+m+1}$. Assuming that $\left.F_{i}\right|_{I_{i}^{0}}=f^{s}$, there exists $0 \leqslant p \leqslant m$ such that $m^{\prime}+s p=$ $m_{N}=n_{i}$. We can rewrite the sum

$$
\begin{aligned}
\sum_{k=1}^{n_{i}-n_{i+m+1}}\left|f^{k+n_{i+m+1}}(T)\right|= & \sum_{j=0}^{N-2} \sum_{k=1}^{m_{j+1}-m_{j}}\left|f^{k+m_{j}}(T)\right|+\sum_{k=1}^{m^{\prime}-m_{N-1}}\left|f^{k+m_{N-1}}(T)\right| \\
& +\sum_{r=0}^{p-1} \sum_{k=1}^{s}\left|f^{k+r s+m^{\prime}}(T)\right| .
\end{aligned}
$$

Using the method from the proof of Proposition 1.3,

$$
\sum_{k=1}^{m^{\prime}}\left|f^{k+m_{N-1}}(T)\right|+\sum_{r=0}^{p-1} \sum_{k=1}^{s}\left|f^{k+r s+m^{\prime}}(T)\right| \leqslant C(\chi) \frac{\sigma_{i, m}}{\left|I_{i}\right|} \sum_{r=0}^{p}\left|f^{r s+m^{\prime}}(T)\right| .
$$

We will deal with the sum on the right hand side later. We will first show that $\sum_{j=0}^{N-2} \sum_{k=1}^{m_{j+1}-m_{j}}\left|f^{k+m_{j}}(T)\right| \leqslant C \sigma_{i, m} \frac{\left|f^{m^{\prime}}(T)\right|}{\left|I_{i}\right|}$.

We denote the left and right components of $I_{j} \backslash I_{j+1}$ by $L_{j}$ and $R_{j}$ respectively. We know from Theorem 2.3(a) and (b) that $\frac{\left|L_{i}\right|}{\left|I_{i+1}\right|}, \frac{\left|R_{i}\right|}{\left|I_{i+1}\right|}>\hat{\chi}$.

We define $\hat{F}_{i}: \bigcup_{j} \hat{I}_{i}^{j} \rightarrow I_{i} \backslash I_{i+1}$ to be the first return map to $I_{i} \backslash I_{i+1}$, such that $\hat{F}_{i}\left(\hat{I}_{i}^{j}\right) \in\left\{L_{i}, R_{i}\right\}$. As in the well bounded case, for each $1 \leqslant j \leqslant N-2$ and $1 \leqslant k<m_{j+1}-m_{j}$, there exists a first entry domain $\hat{U}$ to $I_{i} \backslash I_{i+1}$ such that $f^{k+m_{j}}(T) \subset \hat{U}$. We may assume that $f^{m_{j+1}-m_{j}-k}(\hat{U})=L_{i}$. Indeed, for $1 \leqslant j \leqslant N-3$ there exists $\hat{I}_{i}^{l}$ such that $f^{m_{j+1}}(T) \subset \hat{I}_{i}^{l} \subset L_{i}$. We show that $\hat{I}_{i}^{l}$ is well inside $L_{i}$, which will allow us to estimate $\frac{\left|f^{k+m_{j}}(T)\right|}{|\hat{U}|}$.

Suppose that $\left.F_{i}\right|_{\hat{I}_{i}^{l}}=f^{i_{l}}$. Then there exists an extension to $V_{i}^{l} \supset \hat{I}_{i}^{l}$ such that $f^{i_{l}}: V_{i}^{l} \rightarrow I_{i-1}$. Clearly $V_{i}^{l} \subset L_{i}$, otherwise niceness is contradicted. By Theorems 2.3(a) and 2.4(b), $V_{i}^{l}$ (and thus $L_{i}$ ) is a $\tilde{\chi}$-scaled neighbourhood of $\hat{I}_{i}^{l}$.

For $1 \leqslant j \leqslant N-2$, we have $B\left(L_{i}, f^{m_{j+1}}(T)\right)>\mu B\left(\hat{U}, f^{k+m_{j}}(T)\right)$ where $\mu$ is defined in (4). Therefore,

$$
\left|f^{k+m_{j}}(T)\right|<\frac{|\hat{U}|}{1+\frac{\mu}{B\left(L_{i}, f^{\left.\pi_{j+1}(T)\right)}\right.}}
$$

As in the well bounded case, using a small adaptation of Lemma 3.3, replacing $F_{i}$ by $\hat{F}_{i}$, we can show that $B\left(L_{i}, f^{m_{j}}(T)\right)<\lambda^{N-1-j} B\left(L_{i}, f^{m_{N-1}}(T)\right)$ for $0 \leqslant j \leqslant N-2$. (Note that $\lambda$ is still the $\lambda(\tilde{\chi})$ discussed following Lemma 3.3.) Therefore, it can be shown that

$$
\sum_{j=0}^{N-2} \sum_{k=1}^{m_{j+1}-m_{j}}\left|f^{k+m_{j}}(T)\right| \leqslant \frac{C \sigma_{i, m}}{1-\lambda} B\left(L_{i}, f^{m_{N-1}}(T)\right)
$$

But since $f^{m_{N-1}}(T) \subset I_{i}^{j^{\prime}}$ for some $j^{\prime} \neq 0$, we have

$$
B\left(L_{i}, f^{m_{N-1}}(T)\right)<B\left(L_{i}, I_{i}^{j^{\prime}}\right) \frac{\left|f^{m_{N-1}}(T)\right|}{\left|I_{i}^{j^{\prime}}\right|}
$$

Notice that $F_{i}\left(f^{m_{N-1}}(T)\right)=f^{m^{\prime}}(T)$. So the Koebe Lemma and Lemma 3.1 give $B\left(L_{i}, f^{m_{N-1}}(T)\right)<C(\chi) \Delta \frac{\left|f^{m^{\prime}}(T)\right|}{\left|I_{i}\right|}$, whence

$$
\sum_{j=0}^{N-2} \sum_{k=1}^{m_{j+1}-m_{j}}\left|f^{k+m_{j}}(T)\right| \leqslant C \sigma_{i, m} \frac{\left|f^{m^{\prime}}(T)\right|}{\left|I_{i}\right|}
$$

It remains to bound $\sum_{r=0}^{p}\left|f^{r s+m^{\prime}}(T)\right|^{1+\xi}$ (as can be seen below, we only really need $\xi>0$ for our estimates in the low case). We assume that $f^{m^{\prime}}(T) \cap \partial I_{i+j} \neq \emptyset$ for $1 \leqslant j<m$ : otherwise we have $\sum_{r=0}^{p}\left|f^{r s+m^{\prime}}(T)\right|^{1+\xi}<$ $\left|I_{i}\right|^{1+\xi}$, and we are finished.

Let $\hat{T}=f^{m^{\prime}}(J)$. There exists some $M \geqslant 0$ such that $F_{i}^{M}(\hat{T})=f^{n_{i}}(T)$. We will bound $\sum_{k=0}^{M}\left|F_{i}^{k}(\hat{T})\right|^{1+\xi}$.


Figure 2. When $\hat{T}$ intersects the boundary points $\partial I_{j}$.

If $M$ was uniformly bounded then we would be able to find some bound on $\sum_{k=0}^{M}\left|F_{i}^{k}(\hat{T})\right|$ easily. But $M$ may be very large. We consider this sum in two cases: either $F_{i}$ is high, or $F_{i}$ is low (the high case is the most straightforward). For some background on this dichotomy see $[\mathbf{L y}]$. In both cases, we relabel $\left.F_{i}\right|_{I_{i+1}}$ as $F$ and $I_{i}$ as $I_{0}$. Now let $I_{k}=\left(a_{k}, a_{k}^{\prime}\right)$. We are assuming that $F(c)$ is a maximum for $F$, see Figure 2.

## The high case

We have two cases to consider. We first assume that $F_{j}$ are high and central for $j=0, \ldots, m$. This is known as an Ulam-Neumann cascade.

Lemma 4.1. In the high case, $\sum_{k=0}^{M}\left|F^{k}(\hat{T})\right|<C\left|I_{0}\right|$.

Proof. We know that $I_{0}$ is a $\hat{\chi}$-scaled neighbourhood of $I_{1}$. We will use the Minimum Principle (Theorem 2.5) and Theorem 2.6 to estimate derivatives. The idea here is that either we have derivative uniformly greater than one in $\left(a_{1}, a_{m}\right)$ and we can bound $\sum_{k=0}^{M}\left|F^{k}(\hat{T})\right|$ as a geometric sum; or we have a small derivative in some region, in which case we find a bound on the number of $a_{i}$ that are in this region.

Let $\gamma>1$ satisfy $\frac{\gamma}{\gamma-1}>\frac{1}{2 \hat{\chi}}$. Then we may fix some integer $r \geqslant 1$ such that $2 \hat{\chi} \sum_{i=0}^{r} \gamma^{-i}>1$. Note that $r$ only depends on $\hat{\chi}$. Observe that there is a fixed point $p \in\left(a_{1}, c\right)$. We can choose $I_{0}$ to be so small that the return time to it is greater than $n_{0}$ given in Theorem 2.6. Therefore, by that theorem, $|D F(p)|>\rho_{f}$. If $\left|D F\left(a_{1}\right)\right| \geqslant \gamma$ then from the Minimum Principle, $|D F|_{\left(a_{1}, p\right)}>\gamma^{\prime}$ where $\gamma^{\prime}=\mu^{3} \min \left(\gamma, \rho_{f}\right)$ where $\mu$ is defined in terms of $\left|I_{0}\right|$ in (4). We fix $I_{0}$ to be small enough so that $\gamma^{\prime}>1$. Therefore, we have $\sum_{k=0}^{M}\left|F^{k}(\hat{T})\right|<\frac{\gamma^{\prime}}{\gamma^{\prime}-1}\left|F_{i}^{M}(\hat{T})\right|$.

Suppose now that there is some $u \in\left(a_{1}, c\right)$ such that $|D F|_{\left(a_{1}, u\right)}<\gamma$. We will show that this must mean that $u \in\left(a_{1}, a_{r}\right)$ and thus we can uniformly bound the sum of times that $\hat{T}$ lies in this region.

Suppose that $\left(a_{1}, a_{s}\right) \subset\left(a_{1}, u\right)$. Then we have $\left|a_{i+1}-a_{i}\right|>\frac{\left|a_{i}-a_{i-1}\right|}{\gamma}$ for all $i \leqslant s-1$. Therefore, if $\left(a_{1}, a_{s}\right) \subset U$ then

$$
\left|c-a_{0}\right|>\sum_{i=0}^{s-1}\left|a_{i+1}-a_{i}\right|>\left|a_{1}-a_{0}\right| \sum_{i=0}^{s} \gamma^{-i} .
$$

We know that $\left|a_{1}-a_{0}\right|>2 \hat{\chi}\left|c-a_{0}\right|$. By the definition of $\gamma$ we must have $s \leqslant r$. Moreover, we have $|D F|_{\left(a_{s}, p\right)}>\gamma^{\prime}$.

This helps us bound $\sum_{k=0}^{M}\left|F^{k}(\hat{T})\right|$ where $F^{k}(\hat{T}) \subset I_{0} \backslash I_{m}$. We suppose that $F^{M}(\hat{T})=\left(a_{0}, a_{t}\right)$ for $t \leqslant m$. See Figure 2. Then

$$
\begin{aligned}
\sum_{k=0}^{M}\left|F^{k}(\hat{T})\right|= & \left|a_{1}-a_{0}\right|+\min (2, M-1)\left|a_{2}-a_{1}\right|+\cdots \\
& +\min (i, M-(i-1))\left|a_{i}-a_{t-i}\right|+\cdots+\left|a_{M+t}-a_{M+t-1}\right|
\end{aligned}
$$

This is bounded above by

$$
r\left|a_{r}-a_{0}\right|+\left|a_{N}-a_{N+1}\right| \sum_{i=0}^{\infty} \frac{\min (i, M-(i-1))}{\gamma^{\prime i}} .
$$

The first summand is bounded by $r\left|I_{0}\right|$ and the second summand is bounded above by $C\left|a_{N}-a_{N+1}\right|$ for some $C>0$. So we get $\sum_{k=0}^{M}\left|F_{i}^{k}(\hat{T})\right|<C\left|I_{0}\right|$ as required.

## The low case

We assume that we are in the same setting as above, but with $F_{0}$ central and low. This is known as a saddle node cascade. Again we would like to bound $\sum_{k=0}^{M}\left|F^{k}(\hat{T})\right|$ defined as above. However, as we shall see, we are only able to bound $\sum_{k=0}^{M}\left|F^{k}(\hat{T})\right|^{1+\xi}$.
Lemma 4.2. In the low case, $\sum_{k=0}^{M}\left|F^{k}(\hat{T})\right|^{1+\xi}<C\left|I_{0}\right|^{1+\xi}$.

Proof. We will apply the following result, a form of the Yoccoz Lemma, see for example [FM].
Lemma 4.3. Suppose that $f \in N F^{2}$. Then for all $\delta, \delta^{\prime}>0$ there exists $C>0$ such that if $I_{0}$ is a nice interval such that
(1) $I_{0}$ is a $\delta$-scaled neighbourhood of $I_{1}$;
(2) $F_{k}$ is low and central for $k=0, \ldots, m$;
(3) there is some $0<k<m$ with $\frac{\left|I_{k}\right|}{\left|I_{k+1}\right|}<1+\delta^{\prime}$,
then for $1 \leqslant k<m$,

$$
\frac{1}{C} \frac{1}{\min (k, m-k)^{2}}<\frac{\left|I_{k-1} \backslash I_{k}\right|}{\left|I_{0}\right|}<\frac{C}{\min (k, m-k)^{2}} .
$$

This lemma was suggested by Weixiao Shen. For the proof, see the appendix. (For comparison with other statements of the Yoccoz Lemma, note that we will prove that one consequence of our conditions for the lemma is that we have a lower bound on $\frac{\left|I_{m} \backslash I_{m+1}\right|}{\left|I_{0}\right|}$.)

Suppose that $I_{0}$ satisfies all the conditions of Lemma 4.3. In particular we assume that for some fixed $\delta^{\prime}>0$, we have $\frac{\left|I_{k}\right|}{\left|I_{k+1}\right|}<1+\delta^{\prime}$ for some $0<k<m$. Then for any $\xi>0$,

$$
\begin{aligned}
& \sum_{k=0}^{M}\left|F^{k}(\hat{T})\right|^{1+\xi} \\
& \quad<\sum_{k=0}^{m}\left(\frac{C\left|I_{0}\right|}{\min (k+t, m-(k+t))^{2}}+\cdots+\frac{C\left|I_{0}\right|}{\min (k+1, m-(k+1))^{2}}\right)^{1+\xi} \\
& \quad<C\left|I_{0}\right|^{1+\xi} \sum_{k=0}^{m}\left(\frac{1}{k+1}-\frac{1}{k+t}\right)^{1+\xi} .
\end{aligned}
$$

The sum above is bounded above for any $\xi>0$.
Next we suppose that the hypotheses of Lemma 4.3 do not hold. In particular, this means $\frac{\left|I_{k}\right|}{\left|I_{k+1}\right|} \geqslant 1+\delta^{\prime}$ for $k=0, \ldots, m$. Note that $\left|I_{0}\right| \geqslant\left(1+\delta^{\prime}\right)\left|I_{1}\right| \geqslant$ $\left(1+\delta^{\prime}\right)^{2}\left|I_{2}\right| \geqslant \cdots \geqslant\left(1+\delta^{\prime}\right)^{M}\left|I_{M}\right|$. Therefore

$$
\sum_{k=0}^{M}\left|F^{k}(\hat{T})\right|<\frac{1}{2} \sum_{k=0}^{M} k\left|I_{k}\right| \leqslant \frac{\left|I_{0}\right|}{2} \sum_{k=0}^{M} \frac{k}{\left(1+\delta^{\prime}\right)^{k}}<C\left|I_{0}\right| .
$$

So the lemma is proved.
We have shown that in both low and high cases we have $\sum_{k=0}^{M}\left|F^{k}(\hat{T})\right|^{1+\xi}<$ $C\left|I_{0}\right|^{1+\xi}$. We may apply the usual method to show that this means that $\sum_{k=1}^{n_{i}-m^{\prime}}\left|f^{k+m^{\prime}}(\hat{T})\right|^{1+\xi}<C \sigma_{i, m} \max _{m^{\prime}<k \leqslant n_{i}}\left|f^{k}(T)\right|^{\xi}$. So there is some $C_{\text {casc }}$ such that

$$
\sum_{k=1}^{n_{i}-n_{i+m+1}}\left|f^{k+n_{i+m+1}}(T)\right|^{1+\xi}<C_{\text {casc }} \sigma_{i, m} \max _{n_{i+m}<k \leqslant n_{i}}\left|f^{k}(T)\right|^{\xi}
$$

as required.

## 5. Exceptional case

In the last section we dealt completely with the saddle node cascade. It is easily shown, for example applying Lemma 5.1 below to all branches, that following a saddle node cascade we have a well bounded case, and so the conclusions of Proposition 1.3 hold. An Ulam-Neumann cascade, however, is not always followed by a well bounded case. We estimate the sum for $F_{i}$ in this alternative case here. Most of the sum is dealt with using the
methods for the well bounded case, but we need some new techniques to deal with two of the branches of $F_{i}$.

We consider the sum for $F_{i}$ where $F_{i-2}$ has a central return and $F_{i-1}$ has a high non-central return. The situation here is only slightly different from the case considered in Section 3, since we can prove that all domains of $F_{i}$ are well inside $I_{i}$, except possibly two. Both of these domains $I_{i}^{j}$ have $\left.F_{i}\right|_{I_{i}^{j}}=\left.F_{i-1}\right|_{I_{i}^{j}}$. We denote the left-hand such interval by $I_{i}^{L}$ and the righthand one by $I_{i}^{R}$, see Figure 3. These are the exceptional domains. If $I_{i-1}$ is


Figure 3. The exceptional case.
a $\hat{\chi}$-scaled neighbourhood of $I_{i}$ then by Theorem 2.3 we know that $I_{i}$ is a $\tilde{\hat{\chi}}$-scaled neighbourhood of both $I_{i}^{L}$ and $I_{i}^{R}$, and we may proceed as in the well bounded case. But this will not always be so if $I_{i-1}$ is at the end of a long Ulam-Neumann cascade. So we will assume that $I_{i-1}$ is not a $\tilde{\hat{\chi}}$-scaled neighbourhood of $I_{i}$. Without loss of generality, we suppose that $F_{i-1}(c)$ is a maximum for $F_{i-1}: I_{i} \rightarrow I_{i-1}$.

We are now ready to begin the proof of Proposition 1.5. The strategy for the proof is as follows.

- Show there is some upper bound on $B\left(I_{i}, I_{i}^{j}\right)$ for $j \neq L, R$.
- State our main result in the proof: Proposition 5.3. We suppose that we have some interval $J \subset I_{i}^{j}$ for $j \neq L, R, 0 ; F_{i}(J), \ldots, F_{i}^{m}(J) \subset$ $I_{i}^{L} \cup I_{i}^{R} ;$ and $F_{i}^{m+1}(J) \subset I_{i}^{j^{\prime}}$ for $j^{\prime} \neq L, R, 0$. Then there exists some $\lambda<1$ such that $B\left(I_{i}, J\right)<\lambda B\left(I_{i}, F_{i}^{m+1}(J)\right)$. Furthermore, $\sum_{k=1}^{m}\left|F_{i}^{k}(J)\right|<B\left(I_{i}, F_{i}^{m+1}(J)\right)\left|I_{i}\right|$. We are then able to prove Proposition 1.5. In the rest of this section we prove Proposition 5.3: essentially we need an upper bound on $\sum_{k=1}^{m}\left|F_{i}^{k}(J)\right|$.
- In Lemma 5.4 we show that there exist an interval $V \subset I_{i}$ and $\gamma>1$ such that

$$
\left|D F_{i}\right|_{\left(I_{i}^{L} \cup I_{i}^{R}\right) \backslash V}>\gamma
$$

This allows us to bound parts of the sum $\sum_{k=1}^{m}\left|F_{i}^{k}(J)\right|$ which lie in $\left(I_{i}^{L} \cup I_{i}^{R}\right) \backslash V$.

- We next focus on $V$. We take first return maps to $V$ and use decay of cross-ratios again to estimate sums of intervals in $V$, see Lemma 5.6. We can then complete the proof of Proposition 5.3

We first show in the following simple lemma that we have uniform bounds on how deep the domains of $F_{i}$ are in $I_{i}$ for all domains except $I_{i}^{L}, I_{i}^{R}$.
Lemma 5.1. In the exceptional case outlined above, if $j \neq L, 0, R$ then $I_{i}$ is a $\tilde{\hat{\chi}}$-scaled neighbourhood of $I_{i}^{j}$.

In fact, a similar result holds for the central domain too by Theorem 2.3, but this is not important for us here. This lemma proves that we can treat the case where $F_{i-2}$ is central and $F_{i-1}$ is low and non-central as a well bounded case.

As we shall see, the proof of this lemma is reminiscent of the cascade case since we follow iterates of intervals along the central branch of some $F_{i^{\prime}}$.

Proof. There exists some maximal $i^{\prime}<i$ such that $F_{i^{\prime}-2}$ is non-central. Then by Theorem 2.3, $I_{i^{\prime}}$ is a $\hat{\chi}$-scaled neighbourhood of $I_{i^{\prime}+1}$.

For $j \neq L, R$ we will find $\left.F_{i}\right|_{I_{i}^{j}}$ as a composition of some branches of $F_{i^{\prime}}$ in order to find some extensions. $\left.F_{i^{\prime}}\right|_{I_{i^{\prime}+1}}$ maps $I_{i}^{j}$ out of $I_{i}$ along the cascade, through the sets $I_{i-1} \backslash I_{i}, I_{i-2} \backslash I_{i-1}$ and so on, until it maps to some interval in $I_{i^{\prime}+1} \backslash I_{i^{\prime}+2}$. Then this interval is mapped into some $I_{i^{\prime}}^{j^{\prime}}$. This then maps back into $I_{i^{\prime}+1}$. The process may be repeated many times before $I_{i}^{j}$ is finally mapped back to $I_{i}$.

So know that $\left.F_{i}\right|_{I_{i}^{j}}$ is a composition of maps as follows. Let $j_{1} \neq 0$ be such that $\left(\left.F_{i^{\prime}}^{i-i^{\prime}}\right|_{I_{i^{\prime}+1}}\right)\left(I_{i}^{j}\right) \subset I_{i^{\prime}}^{j_{1}}$. Let $k_{1}=i-i^{\prime}$. If $\left.F_{i}\right|_{I_{i}^{j}}=\left.\left(\left.F_{i^{\prime}}\right|_{I_{i^{\prime}}}\right)\left(\left.F_{i^{\prime}}^{\left(i-i^{\prime}\right)}\right|_{I_{i^{\prime}+1}}\right)\right|_{I_{i}^{j}}$ then we stop here; we say $r=1$. Otherwise, let $k_{2} \geqslant 0$ be minimal such that $F_{i^{\prime}}^{k_{1}+1+k_{2}}\left(I_{i}^{j}\right) \subset I_{i^{\prime}} \backslash I_{i^{\prime}+1}$. Let $j_{2} \neq 0$ be such that $F_{i^{\prime}}^{k_{1}+1+k_{2}}\left(I_{i}^{j}\right) \subset I_{i^{\prime}}^{j_{2}}$. If $\left.F_{i}\right|_{I_{i}^{j}}=\left.F_{i^{\prime}}^{k_{1}+1+k_{2}+1}\right|_{I_{i}^{j}}$ then we stop here; we say $r=2$. Otherwise, we continue this process until we finally return to $I_{i}$ and obtain $k_{r}$.

Suppose that $r=1$. That is,

$$
\left.F_{i}\right|_{I_{i}^{j}}=\left.F_{i^{\prime}}^{\left(i-i^{\prime}\right)+1}\right|_{I_{i}^{j}} .
$$

Let $U$ denote $F_{i^{\prime}}^{\left(i-i^{\prime}\right)}\left(I_{i}^{j}\right)$ and $U^{\prime}$ denote $I_{i^{\prime}}^{j_{1}}$. Then $F_{i^{\prime}}(U)=I_{i}$ and $F_{i^{\prime}}\left(U^{\prime}\right)=$ $I_{i^{\prime}}$. We know that $I_{i^{\prime}}$ is a $\hat{\chi}$-scaled neighbourhood of $I_{i}$. So if we can show
that, taking the appropriate branch, $\left(\left.F_{i^{\prime}}^{-\left(i-i^{\prime}\right)}\right|_{I_{i^{\prime}+1}}\right)\left(U^{\prime}\right) \subset I_{i}$, we know by Theorem 2.4(b) that $I_{i}$ is a $\tilde{\hat{\chi}}$-scaled neighbourhood of $I_{i}^{j}$ (since all the intervals we are concerned with are disjoint). It is easy to see that for this branch, $\left(\left.F_{i^{\prime}}^{-\left(i-i^{\prime}\right)}\right|_{I_{i^{\prime}+1}}\right)\left(U^{\prime}\right) \subset I_{i}$ by the structure of the saddle node cascade since we have $\left(\left.F_{i^{\prime}}^{-1}\right|_{I_{i^{\prime}+1}}\right)\left(U^{\prime}\right) \subset I_{i^{\prime}+1} \backslash I_{i^{\prime}+2},\left(\left.F_{i^{\prime}}^{-2}\right|_{I_{i^{\prime}+1}}\right)\left(U^{\prime}\right) \subset I_{i^{\prime}+2} \backslash I_{i^{\prime}+3}$ and so on. So the lemma is proved when $r=1$.

In the more general case, where $r>1$ and

$$
\left.F_{i}\right|_{I_{i}^{j}}=\left.F_{i^{\prime}}^{\sum_{l=1}^{r}\left(k_{l}+1\right)}\right|_{I_{i}^{j}}
$$

we may apply the same idea, again using the disjointness of the domains of the first return map, to prove that $I_{i}$ is a $\tilde{\hat{\chi}}$-scaled neighbourhood of $I_{i}^{j}$.

If necessary we adjust $\lambda$ so that $\lambda(\tilde{\tilde{\chi}}) \leqslant \lambda<1$.
By the above, if $I_{i}$ is a $\tilde{\hat{\chi}}$-scaled neighbourhood of $I_{i}^{L}$ and $I_{i}^{R}$ then we can proceed with the method in the well bounded case to prove Proposition 1.5. But this is not generally the case. So for our work here, we may assume that $I_{i}$ is not a $\tilde{\hat{\chi}}$-scaled neighbourhood of $I_{i}^{L}$ or $I_{i}^{R}$, and that some iterate of $J$ enters $I_{i}^{L} \cup I_{i}^{R}$.

Remark 5.2. In the previous sections we had uniform upper bounds on the cross-ratio $B\left(I_{i}, I_{i}^{j}\right)$ for all $j$ and so we obtained estimates on the decay of cross-ratios directly. This was used to estimate the sums of intervals. The problem we often encounter in this section is that sometimes we only get good estimates on how cross-ratios decay and sometimes we only get good estimates for the decay of the sizes of intervals. But these estimates are difficult to marry together directly, so we will have to split up such cases. The process is first described in the proof of Proposition 1.5 and again in the proof of Lemma 5.6. (As we will see later, this splitting scheme deals with the cases where we enter $I_{i}^{L} \cup I_{i}^{R}$ from $I_{i} ; V$ from $I_{i}^{L} \cup I_{i}^{R}$; and $\Lambda$ from V.)

The principal result in this section is the following proposition.
Proposition 5.3. If $J, F_{i}(J), \ldots, F_{i}^{m}(J) \subset I_{i}^{L} \cup I_{i}^{R}$ then
(1) there exists some $0 \leqslant \hat{m}<m$ such that $\sum_{k=0}^{m}\left|F_{i}^{k}(J)\right|<C\left(\left|F_{i}^{m}(J)\right|+\right.$ $\left.\left|F_{i}^{\hat{m}}(J)\right|\right)$;
(2) for some $\lambda<1$ independent of $i$, if $F_{i}^{m+1}(J) \subset I_{i}^{j}, j \neq L, 0, R$ then
(a) $\sum_{k=0}^{m}\left|F_{i}^{k}(J)\right|<C B\left(I_{i}, F_{i}^{m+1}(J)\right)\left|I_{i}\right|$;
(b) letting $J^{\prime}$ be the element of $F_{i}^{-1}(J)$ inside some interval $I^{j^{\prime}}$ for $j^{\prime} \neq L, 0, R$ then we have $B\left(I_{i}, J^{\prime}\right)<\lambda B\left(I_{i}, F_{i}^{m+2}\left(J^{\prime}\right)\right)$.

See Figure 4 for a schematic representation of the situation of this proposition. If necessary we will adjust the $\lambda<1$ we use throughout this paper so that we may assume that the proposition above holds for that $\lambda$.


Figure 4. An illustration of Proposition 5.3.

Proof of Proposition 1.5 assuming Proposition 5.3. As in the proof in the well bounded case, we first show that we are principally concerned with the intervals inside $I_{i}$. Again, the proof of this fact is a slight modified version of the proof in the well bounded case.

Let $n_{i+1}<m_{1}<\cdots<m_{j_{i}}=n_{i}$ be all the integers between $n_{i+1}$ and $n_{i}$ such that $f^{m_{j}}(T) \subset I_{i} \backslash I_{i+1}$ for $j=1, \ldots, j_{i}-1$ and let $m_{0}=n_{i+1}$. Let $F_{i}: \bigcup_{i} U_{i}^{j} \rightarrow I_{i}$ be the first entry map to $I_{i}$. As before, we will decompose the sum $\sum_{i=n_{i+1}+1}^{n_{i}}\left|f^{i}(T)\right|$ as $\sum_{j=0}^{j_{i}-1} \sum_{k=1}^{m_{j+1}-m_{j}}\left|f^{m_{j}+k}(T)\right|$.

Suppose that $f^{m_{j}+1}(T) \subset U_{i}^{j}$ for some $U_{i}^{j}$. Suppose further, that $\left.F_{i}\right|_{U_{i}^{j}}=$ $f^{i_{j}}$. Then there exists an extension to $V_{i}^{j} \supset U_{i}^{j}$ so that $f^{i_{j}}: V_{i}^{j} \rightarrow I_{i^{\prime}-1}$ is a diffeomorphism, where $i^{\prime}$ is defined in the proof of Lemma 5.1. Then we have distortion bounds as usual: $\frac{\left|f^{k}\left(f^{m_{j}+1}(T)\right)\right|}{\left|f^{k}\left(U_{i}^{j}\right)\right|} \leqslant C(\chi) \frac{\left|f^{m_{j}+1}(T)\right|}{\left|I_{i}\right|}$. Thus,

$$
\sum_{k=1}^{m_{j+1}-m_{j}}\left|f^{m_{j}+k}(T)\right|<C(\chi) \sigma_{i} \frac{\left|f^{m_{j+1}}(T)\right|}{\left|I_{i}\right|} .
$$

Therefore, $\sum_{j=n_{i+1}+1}^{n_{i}}\left|f^{i}(T)\right|<C(\chi) \frac{\sigma_{i}}{\left|I_{i}\right|} \sum_{j=1}^{j_{i}}\left|f^{m_{i}}(T)\right|$. I.e. we are principally interested in the sum $\sum_{j=1}^{j_{i}}\left|f^{m_{i}}(T)\right|$, that is $\sum_{k=0}^{j_{i}-1}\left|F_{i}^{k}(\hat{T})\right|$ where $\hat{T}=f^{m_{1}}(T)$. In fact, we focus on bounding $\sum_{k=0}^{j_{i}-2}\left|F_{i}^{k}(\hat{T})\right|$.

We split $\hat{T}, F_{i}(\hat{T}), \ldots, F_{i}^{j_{i}-2}(\hat{T})$ into two groups: one for those intervals outside $I_{i}^{L} \cup I_{i}^{R}$ and one for those inside $I_{i}^{L} \cup I_{i}^{R}$. Suppose that $J$ is an interval such that for some $k \geqslant 0$, we have $F_{i}^{k}(J) \subset I_{i}^{j}$ for some $j \neq L, 0, R$; then $F_{i}^{k+1}(J), F_{i}^{k+2}(J), \ldots, F_{i}^{k^{\prime}}(J) \subset I_{i}^{L} \cup I_{i}^{R}$ for some $k^{\prime}>k$; and finally $F_{i}^{k^{\prime}+1}(J) \subset I_{i}^{j^{\prime}}$ for some $j^{\prime} \neq L, 0, R$. From the last part of Proposition 5.3 we have

$$
B\left(I_{i}, F_{i}^{k}(J)\right)<\lambda B\left(I_{i}, F_{i}^{k^{\prime}+1}(J)\right) .
$$

Therefore, we can bound the sums of intervals which lie in the intervals $I_{i}^{j}$ for all $j \neq L, R$ in a similar manner to that for the well bounded case, independently of those intervals inside $I_{i}^{L} \cup I_{i}^{R}$, as follows.
Given $k \geqslant 0$ such that $F_{i}^{k}(\hat{T}) \subset I_{i}^{j}$ for some $j \neq L, 0, R$ we wish to estimate $\left|F_{i}^{k}(\hat{T})\right|$. Let $0 \leqslant \hat{k} \leqslant j_{i}-2$ be maximal such that $F_{i}^{\hat{k}}(\hat{T}) \subset I_{i}^{j^{\prime}}$ for some $j^{\prime} \neq$ $L, R$. Then we apply Proposition 5.3 repeatedly to obtain $B\left(I_{i}, F_{i}^{k}(\hat{T})\right)<$ $\lambda^{l} B\left(I_{i}, F_{i}^{\hat{k}}(\hat{T})\right)$ for some $l \geqslant 0$. The $l$ counts the number of times that $F_{i}^{k+r}(\hat{T})$ lies outside $I_{i}^{L} \cup I_{i}^{R}$ for $0<r \leqslant \hat{k}$. Then

$$
\left|F_{i}^{k}(\hat{T})\right|<\frac{\left|I_{i}\right|}{1+\frac{2}{\lambda^{k} B\left(I_{i}, F_{i}^{k}(\hat{T})\right)}}
$$

We have two cases. In the first case we have $\hat{k}=j_{i}-2$. Then

$$
\begin{aligned}
B\left(I_{i}, F_{i}^{j_{i}-2}(\hat{T})\right) & <B\left(I_{i}, I_{i}^{j^{\prime}}\right) \frac{\left|F_{i}^{j_{i}-2}(\hat{T})\right|}{\left|I_{i}^{j^{\prime}}\right|}<\Delta(\tilde{\hat{\chi}}) \frac{\left|F_{i}^{j_{i}-2}(\hat{T})\right|}{\left|I_{i}^{j^{\prime}}\right|} \\
& <\Delta(\tilde{\hat{\chi}}) C(\tilde{\hat{\chi}})\left|F_{i}^{j_{i}-1}(\hat{T})\right| .
\end{aligned}
$$

Therefore, $\left|F_{i}^{k}(\hat{T})\right|<C \lambda^{l}\left|F_{i}^{j_{i}-1}(\hat{T})\right|$. This suffices to prove an upper bound of the form $C\left|F_{i}^{j_{i}-1}(\hat{T})\right|$ for the $F_{i}$-iterates of $\hat{T}$ outside $I_{i}^{L} \cup I_{i}^{R}$ in this case. In the second case $\hat{k}<j_{i}-2$. We have

$$
B\left(I_{i}, F_{i}^{\hat{k}}(\hat{T})\right)<B\left(I_{i}, I_{i}^{j^{\prime}}\right) \frac{\left|F_{i}^{\hat{k}}(\hat{T})\right|}{\left|I_{i}^{j^{\prime}}\right|}<\frac{\Delta\left|F_{i}^{\hat{k}}(\hat{T})\right|}{\left|I_{i}^{j^{\prime}}\right|}
$$

Since $\left|F_{i}^{\hat{k}}(\hat{T})\right|<C(\chi)\left|F_{i}^{\hat{k}+1}(\hat{T})\right| \frac{\left|I_{i}^{j^{\prime}}\right|}{\left|I_{i}\right|}$. Therefore, in this case we have a bound of the form $C\left|F_{i}^{\hat{k}+1}(\hat{T})\right|$ for the iterates of $T$ outside $I_{i}^{L} \cup I_{i}^{R}$.
Finally we use the above information about sizes of intervals outside $I_{i}^{L} \cup I_{i}^{R}$ to bound the sums of intervals inside $I_{i}^{L} \cup I_{i}^{R}$ too. In the first case above, we have a bound of the form $C\left|F_{i}^{j_{i}-1}(\hat{T})\right|$ for the iterates of $T$ in $I_{i}^{L} \cup I_{i}^{R}$. In the second case above, we have a bound of the form $C\left(\left|F_{i}^{\hat{k}}(\hat{T})\right|+\left|F_{i}^{\hat{n}}(\hat{T})\right|+\right.$ $\left.\left|F_{i}^{j_{i}-1}(\hat{T})\right|\right)$ for the iterates of $T$ in $I_{i}^{L} \cup I_{i}^{R}$.

So in the worst case we have the bound

$$
C_{e x} \sigma_{i}\left(\frac{\left|f^{n_{i}}(T)\right|}{\left|I_{i}\right|}+\frac{\left|f^{n_{i}, 2}(T)\right|}{\left|I_{i}\right|}+\frac{\left|f^{n_{i}, 3}(T)\right|}{\left|I_{i}\right|}\right)
$$

for the sum $\sum_{k=n_{i+1}+1}^{n_{i}}\left|f^{k}(T)\right|$, as required.
5.1. Proof of Proposition 5.3. Denote the smallest interval containing both $I_{i}^{L}$ and $I_{i}^{R}$ by $I_{i}^{\prime}$. Recall that we are assuming that the critical point is a maximum for $\left.F_{i-1}\right|_{I_{i}^{\prime}}$. (Recall that $\left.F_{i}\right|_{I_{i}^{L} \cup I_{i}^{R}}=\left.F_{i-1}\right|_{I_{i}^{L} \cup I_{i}^{R}}$.) This means
that there is some fixed point $p$ of $F_{i}$ in $I_{i}^{R}$. Clearly, there also exists a point $p^{\prime} \in I_{i}^{L}$ such that $F_{i}\left(p^{\prime}\right)=p$. Let $V:=\left(p^{\prime}, p\right)$.

We outline the proof of Proposition 5.3 as follows. We suppose that some iterate of $J$ enters $V$. Let $0 \leqslant s_{1} \leqslant s_{2} \leqslant s_{3}$ be defined as follows. $F_{i}^{k}(J) \subset$ $I_{i}^{\prime} \backslash V$ for $1 \leqslant k \leqslant s_{1} ; F_{i}^{s_{1}+1}(J) \subset V \cap\left(I_{i}^{L} \cup I_{i}^{R}\right)$; and $F_{i}^{s_{2}+k}(J) \subset I_{i}^{\prime} \backslash V$ for $1 \leqslant k \leqslant s_{3}-s_{2}$. Any sum of the form $\sum_{k=0}^{m}\left|F_{i}^{k}(J)\right|$ can be broken up into blocks consisting of such sums.

The scheme for proving Proposition 5.3 is to firstly to show that $\left|D F_{i}\right|_{I_{i}^{\prime} \backslash V}$ is uniformly large. This is proved in Lemma 5.4 and helps to deal with the sums $\sum_{k=0}^{s_{1}}\left|F_{i}^{k}(J)\right|$ and $\sum_{k=1}^{s_{3}-s_{2}}\left|F_{i}^{s_{2}+k}(J)\right|$. Then we have to prove that we have bounds on the sums of intervals which return to $V$. This, proved in Lemma 5.6, helps to deal with $\sum_{k=1}^{s_{2}-s_{1}}\left|F_{i}^{s_{1}+k}(J)\right|$.

Note that the proof of Proposition 5.3 is the only time in this paper that we use the symmetry of the map (and it is only a simplifying assumption).

Lemma 5.4. There exists some $\gamma>1$ independent of $i$ such that

$$
\left|D F_{i}\right|_{I_{i}^{\prime} \backslash V}>\gamma
$$

Proof. We start by observing as in the last section that $\left|D F_{i}(p)\right|>\rho_{f}$. By symmetry, $\left|D F_{i}\left(p^{\prime}\right)\right|>\rho_{f}$ too. Observe that $I_{i}^{L}$ also contains a fixed point $q$ of $F_{i}$. We have $\left|D F_{i}(q)\right|>\rho_{f}$ too. Furthermore, there exists a point $q^{\prime} \in I_{i}^{R}$ such that $F_{i}\left(q^{\prime}\right)=q$. From symmetry, $\left|D F_{i}\left(q^{\prime}\right)\right|>\rho_{f}$.

We can estimate $\left|D F_{i}\right|_{\left(p, q^{\prime}\right)}$ using the Minimum Principle as follows. We use our $\mu$ given in (4) in place of $\mu_{g}$. Then $\left|D F_{i}\right|_{\left(p, q^{\prime}\right)}>\mu^{3} \rho_{f}$. When $I_{0}$ is small enough, $\mu$ is close to 1 . Thus we may ensure that our intervals are so small that $\left|D F_{i}\right|_{\left(p, q^{\prime}\right)}>\rho$ for some $\rho>1$. (To fix precisely how small our intervals must be, we can, for example, choose $\rho=\sqrt{\rho_{f}}$.) By symmetry, $\left|D F_{i}\right|_{\left(q, p^{\prime}\right)}>\rho$.

We deal with the remaining part of the proof of the lemma by showing that $F_{i}$ has large derivative when $x \in I_{i}^{\prime}$ and either $x<q$ or $x>q^{\prime}$. We use the following consequence of Theorem 2.3 and the Minimum Principle.

Claim. There exists some $\gamma^{\prime}=\gamma^{\prime}(\chi)>1$ such that, denoting $I_{i}^{L}=\left(l^{-}, l^{+}\right)$ and $I_{i}^{R}=\left(r^{-}, r^{+}\right)$, if $I_{0}$ is sufficiently small and $B\left(I_{i}, I_{i}^{L}\right), B\left(I_{i}, I_{i}^{R}\right)$ are sufficiently large then

$$
\left|D F_{i}\right|_{\left(l^{-}, q\right)},\left|D F_{i}\right|_{\left(q^{\prime}, r^{+}\right)}>\gamma^{\prime} .
$$

Proof. Let $\theta:=\frac{1}{2}\left(\frac{\left|I_{i^{\prime}}\right|}{\left|I_{i^{\prime}+1}\right|}-1\right)>\hat{\chi}$ where $i^{\prime}$ is defined in the proof of Lemma 5.1. We suppose that $\left|D F_{i^{\prime}}\right|_{I_{i^{\prime}+1} \backslash I_{i}} \leqslant 1+2 \theta$. Then we prove by
induction that $\frac{\left|I_{i^{\prime}+k}\right|}{\left|I_{i^{\prime}+k+1}\right|} \geqslant 1+2 \theta$ for $0 \leqslant k<i-i^{\prime}$. By construction it is true for $k=0$. We assume that it is true for some $0 \leqslant k<i-i^{\prime}-1$. Then

$$
\begin{aligned}
\frac{\left|I_{i^{\prime}+k+1}\right|}{\left|I_{i^{\prime}+k+2}\right|} & \geqslant \frac{\left|I_{i^{\prime}+k+2}\right|+\left(\sup _{I_{i^{\prime}+1} \backslash I_{i}} \mid D F_{i^{\prime}}\right)^{-1}\left|I_{i^{\prime}+k} \backslash I_{i^{\prime}+k+1}\right|}{\left|I_{i^{\prime}+k+2}\right|} \\
& \geqslant 1+\left(\frac{2 \theta}{1+2 \theta}\right) \frac{\left|I_{i^{\prime}+k+1}\right|}{\left|I_{i^{\prime}+k+2}\right|}
\end{aligned}
$$

Then it is easy to see that $\frac{\left|I_{i^{\prime}+k+1}\right|}{\left|I_{i^{\prime}+k+2}\right|} \geqslant 1+2 \theta$ as required.
In particular, we have proved that $\left|D F_{i^{\prime}}\right|_{I_{i^{\prime}+1} \backslash I_{i}} \leqslant 1+2 \theta$ implies that $I_{i}$ is a $\tilde{\theta}$-scaled neighbourhood of both $I_{i}^{L}$ and $I_{i}^{R}$ : a contradiction (since $\tilde{\theta}>\tilde{\tilde{\chi}}$ ). So there must exist some $x \in I_{i^{\prime}+1} \backslash I_{i}$ such that $\left|D F_{i}(x)\right| \geqslant 1+2 \theta>1+2 \hat{\chi}$. Therefore, by Theorem 2.5 and (1) we have

$$
\left|D F_{i^{\prime}}\right|_{\left(x_{0}, p\right)}>\mu^{3} \min \left(1+2 \hat{\chi}, \rho_{f}\right)
$$

Choosing $\left|I_{0}\right|$ small we have some $\gamma^{\prime}>1$ such that $\left|D F_{i}\right|_{\left(x_{0}, q\right)}>\gamma^{\prime}$. In particular $\left|D F_{i}\right|_{\left(l^{-}, q\right)}>\gamma^{\prime}$. Similarly we can show $\left|D F_{i}\right|_{\left(q^{\prime}, r^{+}\right)}>\gamma^{\prime}$.

Letting $\gamma:=\min \left(\rho, \gamma^{\prime}\right)$, the lemma is proved.
By the above, we will be able to estimate the sizes of iterates of $T$ inside $\left(I_{i}^{L} \cup I_{i}^{R}\right) \backslash V$ as a geometric sum.

We will need some real bounds for $V$. The following lemma, which contrasts with Lemma 5.4, will later be used to obtain these bounds.
Lemma 5.5. There exists some $\hat{C}=\hat{C}\left(\chi,\left|I_{i}^{\prime}\right|\right)>0$, where $\hat{C}\left(\chi,\left|I_{i}^{\prime}\right|\right)$ tends to some constant $\hat{C}(\chi)$ as $\left|I_{i}^{\prime}\right| \rightarrow 0$, such that

$$
\left|D F_{i}\right|_{I_{i}^{L} \cup I_{i}^{R}}<\hat{C}
$$

Proof. We work with $F_{i^{\prime}}: I_{i^{\prime}+1} \rightarrow I_{i^{\prime}}$ where $i^{\prime}$ is defined in the proof of Lemma 5.1. There exists some $m \geqslant 1$ such that $\left.F_{i^{\prime}}\right|_{I_{i^{\prime}+1}}=\left.f^{m}\right|_{I_{i^{\prime}+1}}$. We can decompose this map into two maps so that $F_{i^{\prime}}=L \circ g$ where $g=\left.f\right|_{U_{\phi}}$, i.e $g(x)=f(c)-|\phi(x)|^{\alpha}$, and $L=f^{m-1}: f\left(I_{i^{\prime}+1}\right) \rightarrow I_{i^{\prime}}$.
By Theorems 2.4(a) and 2.3(a) we have $\frac{D L(x)}{D L(y)}<C(\chi)$ for $x, y \in f\left(I_{i+1}\right)$. So

$$
|D L(x)| \leqslant C(\chi) \frac{\left|I_{i}\right|}{\left|f\left(I_{i+1}\right)\right|}=C(\chi) \frac{\left|I_{i}\right|}{\left|\phi\left(\frac{\left|I_{i+1}\right|}{2}\right)^{\alpha}\right|}
$$

for $x \in f\left(I_{i+1}\right)$. Also

$$
|D g(x)|=\alpha|D \phi(x)|\left|\phi(x)^{\alpha-1}\right|<\alpha \sup _{x \in I_{i^{\prime}+1}}|D \phi(x)|\left|\phi\left(\frac{\left|I_{i+1}\right|}{2}\right)\right|^{\alpha-1} .
$$

For $\hat{U} \subset U_{\phi}$ a small neighbourhood of $c$, let $\operatorname{Dist}(\phi, \hat{U}):=\sup _{x, y \in \hat{U}} \frac{|D \phi(x)|}{|D \phi(y)|}$. Observe that as $I_{i}^{\prime}$ becomes smaller, $\operatorname{Dist}\left(\phi, I_{i}^{\prime}\right)$ tends to 1 . For $x \in I_{i}^{L} \cup I_{i}^{R}$,

$$
\left|D F_{i}(x)\right|<\alpha C(\chi) \frac{\sup _{x \in I_{i+1}}|D \phi(x)|\left|I_{i}\right|}{\left|\phi\left(\frac{\left|I_{i+1}\right|}{2}\right)\right|}<2 \alpha C(\chi) \operatorname{Dist}\left(\phi, I_{i}^{\prime}\right) \frac{\left|I_{i}\right|}{\left|I_{i+1}\right|}
$$

Since we have assumed that $\frac{\left|I_{i}\right|}{\left|I_{i+1}\right|}$ is bounded below, there is some constant $C>0$ such that for all $x \in I_{i}^{\prime}$,

$$
\left|D F_{i}(x)\right|<C C(\chi) \operatorname{Dist}\left(\phi, I_{i}^{\prime}\right)
$$

Letting $\hat{C}\left(\chi,\left|I_{i}^{\prime}\right|\right):=C C(\chi) \operatorname{Dist}\left(\phi, I_{i}^{\prime}\right)$ we have proved the lemma.
We denote the first return map to $V$ by $\hat{F}_{i}: \bigcup_{j} V^{j} \rightarrow V$. We first wish to find some control on the sizes of the domains of $\hat{F}_{i}$. Let $m_{V, j}$ be such that $\left.\hat{F}_{i}\right|_{V^{j}}=\left.F_{i}^{m_{V, j}}\right|_{V^{j}}$. The following lemma is key to proving Proposition 5.3.

Lemma 5.6. If $F_{i}^{l_{1}}(J), \ldots, F_{i}^{l_{m}}(J) \subset V \cap\left(I_{i}^{L} \cup I_{i}^{R}\right)$ are all the iterates of $J$ up to $l_{m}$ which lie in $V \cap\left(I_{i}^{L} \cup I_{i}^{R}\right)$, and all intermediate iterates $F_{i}^{k}(J)$ for $k=0,1, \ldots, l_{m}$ lie in $I_{i}^{L} \cup I_{i}^{R}$ then

$$
\sum_{k=0}^{l_{m}}\left|F_{i}^{k}(J)\right|<C\left|F_{i}^{l_{m}}(J)\right| .
$$

Furthermore, there exists $\lambda_{V}<1$ such that $|J|<C \lambda_{V}^{l_{m}-m}\left|F_{i}^{l_{m}}(J)\right|$.
Proof. We split the sum as follows

$$
\sum_{k=0}^{l_{m}}\left|F_{i}^{k}(J)\right|=\sum_{j=0}^{m-1} \sum_{k=1}^{l_{j+1}-l_{j}}\left|F_{i}^{l_{j}+k}(J)\right|
$$

where we let $l_{0}=-1$. We know from Lemma 5.4 that $\left|D F_{i}\right|_{I_{i}^{\prime} \backslash V}>\gamma$ so

$$
\sum_{k=1}^{l_{j+1}-l_{j}}\left|F_{i}^{l_{j}+k}(J)\right|<\left|F_{i}^{l_{j+1}}(J)\right| \sum_{k=0}^{l_{j+1}-l_{j}-1} \gamma^{-k}<\frac{\left|F_{i}^{l_{j+1}}(J)\right|}{1-\gamma^{-1}} .
$$

Whence,

$$
\sum_{k=0}^{l_{m}}\left|F_{i}^{k}(J)\right|<\frac{1}{1-\gamma^{-1}} \sum_{j=0}^{m}\left|F_{i}^{l_{j}}(J)\right| .
$$

So we only need bound the sum of returns to $V$.
Denote the rightmost element of $\bigcup^{j} V^{j}$ by $V^{1}$ and the leftmost element by $V^{2}$ (observe that $\left.\hat{F}_{i}\right|_{V^{1}}=\left.F_{i}^{2}\right|_{V^{1}}$ and $\left.\hat{F}_{i}\right|_{V^{2}}=\left.F_{i}^{2}\right|_{V^{2}}$ ). We get an estimate on how deep each $V^{j}$ is inside $V$ for $j>2$ because $V^{1}$ and $V^{2}$ have some definite size compared to $|V|$; since by Lemma 5.5 we know that $\left|V^{1}\right|,\left|V^{2}\right|>$ $\frac{|V|}{\hat{C}^{2}}$. Therefore, there exists some $\delta_{0}^{\prime}$ depending only on $f$ such that $V$ is a $\delta_{0}^{\prime}$-scaled neighbourhood of $V^{j}$ for all $j>2$. So by Lemma 3.2, there exists some $\lambda_{V}^{\prime}<1$ depending on $\delta_{0}^{\prime}$ such that for any interval $J^{\prime} \subset V^{j}$,
$B\left(V, J^{\prime}\right)<\lambda_{V}^{\prime} B\left(V^{j}, J^{\prime}\right)$ for $j>2$ (in fact this is also shown in the claim below). As usual we can use Lemma 3.3 to conclude that there exists some $\lambda_{V}<1$ such that $B\left(V, J^{\prime}\right)<\lambda_{V} B\left(V, \hat{F}_{i}\left(J^{\prime}\right)\right)$. If we remain away from $V^{1}$ and $V^{2}$, this fact and the usual argument would be sufficient to obtain the required bound on sums.

We must deal with the case where iterates enter $V^{1}, V^{2}$. The idea is to split the situation into the case where intervals land in a region where $\left|D \hat{F}_{i}\right|$ is large and the case when the intervals land in a region where we don't have good estimates on $\left|D \hat{F}_{i}\right|$.

We first focus on $V^{2}$. We know from Theorem 2.6 that $\left|D F_{i}\left(p^{\prime}\right)\right|>\rho_{f}$ and so $\left|D \hat{F}_{i}\left(p^{\prime}\right)\right|>\rho_{f}^{2}$. There must also exist some fixed point $r$ of $\hat{F}_{i}$ in $V^{2}$ with $\left|D \hat{F}_{i}(r)\right|>\rho_{f}$. Letting $\Lambda_{2}:=\left(p^{\prime}, r\right)$ and applying the Minimum Principle as before, we obtain $\left|D \hat{F}_{i}\right|_{\Lambda_{2}}>\rho$ for some $\rho>1$. Let $r^{\prime}$ be the point in $V^{1}$ such that $\hat{F}_{i}\left(r^{\prime}\right)=r$. Then adjusting $\rho>1$ if necessary, $\left|D \hat{F}_{i}\right|_{\left(r^{\prime}, p\right)}>\rho$. We define $\Lambda_{1}$ to be the interval in $V^{1}$ which has $\hat{F}_{i}\left(\Lambda_{1}\right)=V \backslash V^{2}$. Clearly $\Lambda_{1} \subset\left(r^{\prime}, p\right)$, so $\left|D \hat{F}_{i}\right|_{\Lambda_{1}}>\rho$. For convenience later, we let $\Lambda:=\Lambda_{1} \cup \Lambda_{2}$.
We are now ready to deal with bounding $\sum_{k=0}^{m-1}\left|\hat{F}_{i}^{k}(J)\right|$. Observe that $\hat{F}_{i}{ }^{m-1}(J)$ must be contained in some $V^{j}$. Suppose first that $j>2$; we deal with the case where $j=1$ or 2 later. Suppose further that $J \subset V^{j^{\prime}}$ and $j^{\prime}>2$; here the other case is similar. We will again split up the sum. Let $N_{0}^{\prime}=0$. Let $N_{1}$ be minimal such that $\hat{F}_{i}^{N_{1}}(J) \cap \Lambda=\emptyset$ and $\hat{F}_{i}^{N_{1}+1}(J) \subset \Lambda$. Let $N_{1}^{\prime}>N_{1}$ be minimal such that $\hat{F}_{i}^{N_{1}^{\prime}}(J) \subset \Lambda$ and $\hat{F}_{i}^{N_{1}^{\prime}+1}(J) \cap \Lambda=\emptyset$. In this way we obtain $N_{0}^{\prime}<N_{1}<N_{1}^{\prime}<\cdots<N_{M-1}<N_{M-1}^{\prime}$ so that

$$
\begin{aligned}
\sum_{k=0}^{m-1}\left|\hat{F}_{i}^{k}(J)\right|= & \sum_{j=0}^{M-1}\left(\sum_{k=1}^{N_{j+1}-N_{j}^{\prime}}\left|\hat{F}_{i}^{N_{j}^{\prime}+k}(J)\right|+\sum_{k=1}^{N_{j+1}^{\prime}-N_{j+1}}\left|\hat{F}_{i}^{N_{j+1}+k}(J)\right|\right) \\
& +\sum_{k=1}^{N_{M}-N_{M-1}^{\prime}}\left|\hat{F}_{i}^{N_{M-1}^{\prime}+k}(J)\right|
\end{aligned}
$$

where $N_{M}=m-1$. Observe that the first sum in the brackets concerns intervals which land inside $\Lambda$ and the second sum in the brackets concerns intervals in $V \backslash \Lambda$. Then

$$
\sum_{k=1}^{N_{j+1}^{\prime}-N_{j+1}}\left|\hat{F}_{i}^{N_{j+1}+k}(J)\right|<\left|\hat{F}_{i}^{N_{j+1}^{\prime}}(J)\right| \sum_{k=0}^{N_{j+1}-N_{j+1}^{\prime}-1} \rho^{-k}<\frac{C}{1-\rho^{-1}}\left|\hat{F}_{i}^{N_{j+1}^{\prime}}(J)\right|
$$

for some $C$.
Now we consider $\sum_{k=1}^{N_{j+1}-N_{j}^{\prime}}\left|\hat{F}_{i}^{N_{j}^{\prime}+k}(J)\right|$. In fact we learn most from estimating the sum $\sum_{k=1}^{N_{M}-N_{M-1}^{\prime}}\left|\hat{F}_{i}^{N_{M-1}^{\prime}+k}(J)\right|$. If necessary we make $\lambda_{V}<1$ smaller so that for $J \subset V^{j} \backslash \Lambda_{j}$ for $j=1,2$ we have $B(V, J)<\lambda_{V} B\left(V, F_{i}(J)\right)$. Then
for $1 \leqslant k<N_{m}-N_{M-1}^{\prime}$,

$$
B\left(V, \hat{F}_{i}^{N_{M-1}^{\prime}+k}(J)\right)<\lambda_{V}^{N_{M}-N_{M-1}^{\prime}-k} B\left(V, \hat{F}_{i}^{N_{M}}(J)\right) .
$$

Recalling that $M=m-1$ we calculate $B\left(V, \hat{F}_{i}^{m-1}(J)\right)<B\left(V, V^{j}\right) \frac{\left|\hat{F}_{i}^{m-1}(J)\right|}{\left|V^{j}\right|}$. Letting $B_{V}:=\max \left\{\sup _{j>2} B\left(V, V^{j}\right), B\left(V, V^{1} \backslash \Lambda_{1}\right), B\left(V, V^{2} \backslash \Lambda_{2}\right)\right\}$, we obtain

$$
\left|\hat{F}_{i}^{N_{M-1}^{\prime}+k}(J)\right|<\frac{|V|}{1+\frac{2\left|V^{j}\right|}{\lambda_{V}^{N_{M}-N_{M-1}^{\prime-k}} B_{V}\left|\hat{F}_{i}^{m-1}(J)\right|}} .
$$

Letting $\hat{B}_{V}:=\frac{B_{V}}{B_{V}+2}$ we have

$$
\left|\hat{F}_{i}^{N_{M-1}^{\prime}+k}(J)\right|<\hat{B}_{V} \lambda_{V}^{N_{M}-N_{M-1}^{\prime}+k} \frac{|V|}{\left|V^{j}\right|}\left|\hat{F}_{i}^{m-1}(J)\right| .
$$

Hence we have

$$
\sum_{k=1}^{N_{j+1}-N_{j}^{\prime}}\left|\hat{F}_{i}^{N_{j}^{\prime}+k}(J)\right|<C\left|\hat{F}_{i}^{m-1}(J)\right| .
$$

We now estimate the other sums concerning intervals outside $\Lambda$ as follows. Let $\mu^{\prime}:=\exp \left\{-\sigma^{\prime}\left(I_{0}\right) \frac{\left|I_{0}\right|}{1-\rho^{-1}}\right\}$. Suppose that $F_{i}^{N_{M-2}}(J) \subset V^{j}$. Then taking the appropriate branch, $\hat{F}_{i}^{N_{M-2}-N_{M-1}^{\prime-1}}(V) \subset V^{j}$ and

$$
\begin{aligned}
B\left(V, \hat{F}_{i}^{N_{M-2}}(J)\right) & <\lambda_{V}^{\prime} B\left(\hat{F}_{i}^{N_{M-2}-N_{M-1}^{\prime}-1}(V), \hat{F}_{i}^{N_{M-2}}(J)\right) \\
& <\frac{\lambda_{V}^{\prime}}{\mu^{\prime}} B\left(\hat{F}_{i}^{-1}(V), \hat{F}_{i}^{N_{M-1}^{\prime}}(J)\right) \\
& <\frac{\lambda_{V}^{\prime}}{\mu \mu^{\prime}} B\left(V, \hat{F}_{i}^{N_{M-1}^{\prime}+1}(J)\right)
\end{aligned}
$$

Shrinking $I_{0}$ if necessary, as usual, so that $\frac{\lambda_{V}^{\prime}}{\mu \mu^{\prime}}=: \lambda_{V}<1$, we obtain

$$
B\left(V, \hat{F}_{i}^{N_{M-2}}(J)\right)<\lambda_{V} B\left(V, \hat{F}_{i}^{N_{M-1}^{\prime}+1}(J)\right)
$$

Clearly then we can proceed in bounding the sum using the usual method of decaying cross-ratios. So can bound $\sum_{k=0}^{m-1}\left|\hat{F}_{i}^{k}(J)\right|$ above by $C\left|\hat{F}_{i}^{m-1}(J)\right|$ for this case.

To complete this case, we will bound $\left|\hat{F}_{i}^{m-1}(J)\right|$ in terms of $\left|\hat{F}_{i}^{m}(J)\right|$. We do this by constructing an extension. Let the left-hand and right-hand members of $F_{i}^{-1}\left(p^{\prime}\right)$ be denoted by $b$ and $b^{\prime}$ respectively. Denote $\left(b, b^{\prime}\right)$ by $V^{\prime}$. By Lemma 5.5, $V^{\prime}$ is a $\delta_{V^{\prime}}$-scaled neighbourhood of $V$ where $\delta_{V^{\prime}}$ depends only on $f$.

Claim. For all domains $V^{j}, j>2$ there exists an extension to some interval $U^{j} \supset V^{j}$ such that $U^{j} \subset V$ and $F_{i}^{m_{V, j}}: U^{j} \rightarrow V^{\prime}$ is a diffeomorphism.

Proof. For $j>2$ the return maps are a composition of $\left.F_{i}\right|_{V}$ followed by $\left.F_{i}\right|_{I_{i}^{R}}$ and then some number of iterates of $\left.F_{i}\right|_{I_{i}^{L}}$. So $\hat{F}_{i}^{-1}$ must pull $V^{\prime}$ back into $I_{i}^{L}$. Observe that this element of $F_{i}^{-1}\left(V^{\prime}\right)$ is below $p^{\prime}$ (and clearly away from $\left.F_{i}(c)\right)$. Any further pullbacks in $I_{i}^{L}$ remain below $p^{\prime}$ also. Therefore when some element $F_{i}^{-k}\left(V^{\prime}\right)$ is finally pulled back into $I_{i}^{R}$, it is mapped above $p$ and remains away from $F_{i}(c)$. Therefore we have elements of $F_{i}^{-k-2}\left(V^{\prime}\right)$ mapping inside $V$ which don't contain $c$.

By the above claim and Theorem 2.4 we have some $C>0$ depending only on $f$ such that if $j>2$,

$$
\frac{1}{C} \frac{|V|}{\left|V^{j}\right|} \leqslant\left|D \hat{F}_{i}\right|_{V^{j}} \leqslant C \frac{|V|}{\left|V^{j}\right|}
$$

(Recall that we are assuming that $F_{i}^{m-1}(V) \cap \Lambda=\emptyset$.)
Therefore,

$$
\sum_{k=1}^{N_{M}-N_{M-1}^{\prime}}\left|\hat{F}_{i}^{N_{M-1}^{\prime}+k}(J)\right|<C\left|\hat{F}_{i}^{m}(J)\right|
$$

There remains a further case to consider. Above we assumed $\hat{F}_{i}^{m-1}(J) \subset V^{j}$ where $j>2$. But if $j \in\{1,2\}$ we have two cases. We first note that if $F_{i}^{l_{m}}(J) \cap\left\{r, r^{\prime}\right\}=\emptyset$ then the intervals we are concerned with are either completely inside $\Lambda_{2}, \Lambda_{1}$ or completely inside $V \backslash\left(\Lambda_{2} \cup \Lambda_{1}\right)$. Then we may proceed as above. But if $F_{i}^{k}(J)$ contains $r$ or $r^{\prime}$ then we split $F_{i}^{k}(J)$ into two intervals, with this periodic point at their intersection. We may then apply the procedure above to estimate the size of each interval. We need only apply this splitting argument once since if we intersect a periodic point of $\hat{F}_{i}$ once, we must stay there for all time under iteration by $\hat{F}_{i}$. Thus we need only alter our constants by a factor of 2 to deal with this case. Note that we only have one sum where this problem could occur: $\sum_{k=1}^{N_{M}^{\prime}-N_{M}}\left|\hat{F}_{i}^{N_{M}+k}(J)\right|$ where $N_{M}^{\prime}=m$. This is because $r$ is a fixed point for $\hat{F}_{i}$.

Clearly, we can use the cross-ratio argument as usual to obtain the estimate $\left|F_{i}^{l_{1}}(J)\right|<\lambda_{V}^{m-1} C\left|F_{i}^{l_{m}}(J)\right|$, so $|J|<\lambda_{V}^{m-1} C\left|F_{i}^{l_{m}}(J)\right|$.

We may adjust our usual $\lambda$ so that $\lambda_{V} \leqslant \lambda<1$.

Proof of Proposition 5.3. Suppose first that $F_{i}^{m+1}(J) \subset I_{i}^{j}$ for $j \neq L, R$. Then, in particular, we can be sure that $F_{i}^{m}(J)$ does not contain $p$ or $p^{\prime}$. Then we also know that none of $F_{i}^{k}(J)$ contain $p$ or $p^{\prime}$ for $0 \leqslant k \leqslant m-1$. This means that we can be sure that all the intervals we consider are either contained in $V$ or are disjoint from $V$.

Recall that $0 \leqslant s_{1}<s_{2} \leqslant s_{3}=m$ are defined as follows. (We suppose that some iterate of $J$ enters $V$ : otherwise the proof is simpler.) $F_{i}^{k}(J) \subset I_{i}^{\prime} \backslash V$ for $1 \leqslant k \leqslant s_{1} ; F_{i}^{s_{1}+1}(J) \subset V \cap\left(I_{i}^{L} \cup I_{i}^{R}\right) ;$ and $F_{i}^{s_{2}}(J) \subset V \cap\left(I_{i}^{L} \cup I_{i}^{R}\right)$, $F_{i}^{s_{2}+k}(J) \subset I_{i}^{\prime} \backslash V$ for $1 \leqslant k \leqslant s_{3}-s_{2}$.

Then if $s_{3}>s_{2}$,

$$
\sum_{k=1}^{s_{3}-s_{2}}\left|F_{i}^{s_{2}+k}(J)\right|<\left|F_{i}^{s_{3}}(J)\right| \sum_{k=0}^{s_{3}-s_{2}-1} \gamma^{-k}<C\left|F_{i}^{s_{3}}(J)\right|,
$$

by Lemma 5.4.
From Lemma 5.6,

$$
\sum_{k=1}^{s_{2}-s_{1}}\left|F_{i}^{s_{1}+k}(J)\right|<C\left|F_{i}^{s_{2}}(J)\right|
$$

and $\left|F_{i}^{s_{1}+1}(J)\right|<C\left|F_{i}^{s_{2}}(J)\right|$.
Also

$$
\sum_{k=0}^{s_{1}}\left|F_{i}^{k}(J)\right|<\gamma^{-1}\left|F_{i}^{s_{1}+1}(J)\right| \sum_{k=0}^{s_{1}} \gamma^{-k}<C\left|F_{i}^{s_{2}}(J)\right| .
$$

Therefore,

$$
\sum_{k=0}^{s_{2}}\left|F_{i}^{k}(J)\right|<C\left|F_{i}^{s_{2}}(J)\right|
$$

If $s_{3}>s_{2}$ then

$$
\sum_{k=0}^{s_{3}}\left|F_{i}^{k}(J)\right|<C\left(\left|F_{i}^{s_{3}}(J)\right|+\left|F_{i}^{s_{2}}(J)\right|\right) .
$$

Therefore, the first part of the proposition is proved.
Now if $F_{i}^{m+1}(J) \subset I_{i}^{j}$ for $j \neq L, R, 0$ then recalling that $s_{3}=m$ we will obtain an estimate for $\left|F_{i}^{s_{2}}(J)\right|$ in terms of $B\left(I_{i}, F_{i}^{m+1}(J)\right)$.
$B\left(I_{i}, F_{i}^{s_{2}}(J)\right)<B\left(F_{i}^{-s_{3}+s_{2}}\left(I_{i}\right), F_{i}^{s_{2}}(J)\right)<\frac{B\left(I_{i}, F_{i}^{m}(J)\right)}{\mu}<\frac{B\left(I_{i}, F_{i}^{m+1}(J)\right)}{\mu^{2}}$.
We are allowed to use $\mu$ here since all intermediate intervals must be disjoint (otherwise we would have to pass through $V$ again). Therefore

$$
\left|F_{i}^{s_{2}}(J)\right|<\frac{\left|I_{i}\right|}{1+\frac{2 \mu^{2}}{B\left(I_{i}, F_{i}^{m+1}(J)\right)}}<C\left|I_{i}\right| B\left(I_{i}, F_{i}^{m+1}(J)\right) .
$$

Similarly we can show that $\left|F_{i}^{m}(J)\right|<C\left|I_{i}\right| B\left(I_{i}, F_{i}^{m+1}(J)\right)$. Therefore

$$
\sum_{k=0}^{s_{3}}\left|F_{i}^{k}(J)\right|<C\left|I_{i}\right| B\left(I_{i}, F_{i}^{m+1}(J)\right)<C_{1}\left|I_{i}\right|
$$

for some $C_{1}>0$.

We now prove the final part of the proposition. Clearly for any run of intervals $F_{i}(J), \ldots, F_{i}^{k}(J) \subset I_{i}^{L} \cup I_{i}^{R}$, considering the branch of $F_{i}^{-k}$ which follows the iterates of $J$, we have $B\left(F_{i}^{k}, F_{i}^{-k}\left(I_{i}\right), J\right)>\mu^{\prime \prime}$ where $\mu^{\prime \prime}:=$ $\exp \left\{-C_{1} \sigma^{\prime}\left(\left|I_{0}\right|\right)\left|I_{0}\right|\right\}$. We consider the branch of $F_{i}^{-m-2}$ which follows the backward orbit of $F_{i}^{m+1}(J)$. Clearly, $F_{i}^{-m-2}\left(I_{i}\right)$ is strictly inside $I_{i}^{j}$. Thus,

$$
\begin{aligned}
B\left(I_{i}, J^{\prime}\right) & <\lambda^{\prime} B\left(I_{i}^{j}, J^{\prime}\right)<\lambda^{\prime} B\left(F_{i}^{-m-2}\left(I_{i}\right), J^{\prime}\right)<\frac{\lambda^{\prime}}{\mu^{\prime \prime}} B\left(F_{i}^{-1}\left(I_{i}\right), F_{i}^{m+1}\left(J^{\prime}\right)\right) \\
& <\frac{\lambda^{\prime}}{\mu^{\prime \prime} \mu} B\left(I_{i}, F_{i}^{m+2}\left(J^{\prime}\right)\right) .
\end{aligned}
$$

For $\left|I_{0}\right|$ small enough, we can alter the usual $\lambda$ slightly so that $\frac{\lambda^{\prime}}{\mu^{\prime \prime} \mu} \leqslant \lambda$ and still ensure that $\lambda<1$. Thus, $B\left(I_{i}, J^{\prime}\right)<\lambda B\left(I_{i}, F_{i}^{m+2}\left(J^{\prime}\right)\right)$ as required.

When we do not escape $I_{i}^{L} \cup I_{i}^{R}$ then we may have some intersection with $p$ or $p^{\prime}$. In this case, we split our interval in two and estimate the size of each piece as above. We need only apply this idea once, so we can change our constants to cater for this case too. In this case, part 2 of the proposition doesn't occur.

## 6. PROOF OF THE MAIN THEOREM IN THE NON-INFINITELY RENORMALISABLE CASE

We recall that $B\left(f^{n}, T, J\right)>\exp \left\{-C \sum_{k=0}^{n-1}\left|f^{k}(T)\right|^{1+\eta}\right\}$ when $f \in C^{2+\eta}$. We will find a bound on the sum $\sum_{k=0}^{n-1}\left|f^{k}(T)\right|^{1+\eta}$ by using the main propositions above and also finding some decay property for the size of the domains of $F_{i}$ for some values of $i$. We assume that $f^{k}(T) \cap \partial I_{j} \neq \emptyset$ only within a cascade case (i.e. when there exist $i, m$ such that $F_{i}$ is in a cascade case and $\left.f^{k}(T) \subset I_{i} \backslash I_{i+m}\right)$. It is easy to see how to extend the proof when this is not true.

Let $F_{i}: \cup_{j} U_{i}^{j} \rightarrow I_{i}$ be the first entry map to $I_{i}$ (we include the branches of the first return map in this case too). For $i<j$ and an interval $V$, we define $S(i, j, V)$ to be the maximum of $\left|f^{i+1}(V)\right|,\left|f^{i+2}(V)\right|, \ldots,\left|f^{j}(V)\right|$. We will consider $S\left(n_{i+1}, n_{i}, T\right)$. Let $n(i, j)$ be such that $\left.F_{i}\right|_{U_{i}^{j}}=\left.f^{n(i, j)}\right|_{U_{i}^{j}}$. Now let $U_{i}^{s(i)}$ be the interval for which $S\left(0, n(i, j), U_{i}^{j}\right)$ is maximal. Let $\hat{n}(i)=n(i, s(i))$. Clearly,

$$
S\left(n_{i+1}, n_{i}, T\right) \leqslant S\left(0, \hat{n}(i), U_{i}^{s(i)}\right) .
$$

We would like to show that for certain $i$, this quantity decays with $i$ in a controlled way.

We start by assuming that $F_{i-1}$ is in a well bounded case. We have two cases. Firstly, suppose that $U_{i}^{s(i)} \subset I_{i}$. Then since $F_{i-1}$ is in a well bounded
case, we have $\left|U_{i}^{s(i)}\right|<\frac{\left|I_{i-1}\right|}{1+2 \chi}$. Since $I_{i}$ is a domain of the first return map to $I_{i-1}$ we have

$$
\left|U_{i}^{s(i)}\right|<\frac{S\left(0, \hat{n}(i-1), U_{i-1}^{s(i-1)}\right)}{1+2 \chi}
$$

Now assume that $U_{i}^{s(i)} \cap I_{i}=\emptyset$. Then there exists some extension $V_{i} \supset U_{i}^{s(i)}$ such that $f^{n(s(i))}: V_{i} \rightarrow I_{i-1}$ is a diffeomorphism. We will show that $U_{i}^{s(i)}$ is uniformly smaller than $V_{i}$. By (1) we know that $B\left(V_{i}, U_{i}^{s(i)}\right)<\frac{B\left(I_{i-1}, I_{i}\right)}{\mu}$ for $\mu$ as in (4). Thus, by Lemma 3.1, $\left|U_{i}^{s(i)}\right|<\frac{\left|V_{i}\right|}{1+\frac{2 \mu}{\Delta(x)}}$. Since $V_{i}$ is a first return domain to $I_{i-1}$ we have

$$
\left|U_{i}^{s(i)}\right|<\frac{S\left(0, \hat{n}(i-1), U_{i-1}^{s(i-1)}\right)}{1+\frac{2 \mu}{\Delta(\chi)}}
$$

Let $\gamma:=\max \left(\frac{1}{1+2 \chi}, \frac{1}{1+\frac{2 \mu}{\Delta(\chi)}}\right)$. Clearly $\gamma<1$. So

$$
S\left(0, \hat{n}(i), U_{i}^{s(i)}\right)<\gamma S\left(0, \hat{n}(i-1), U_{i-1}^{s(i-1)}\right)
$$

We let $C_{\text {all }}=\max \left(C_{w b}, C_{\text {casc }}, 3 C_{e x}\right)$. Note that by disjointness, all $\sigma_{i}, \sigma_{i, m}<$ 1. If $f \in N F^{2+\eta}$ and $F_{i-1}$ is well bounded, we have

$$
\begin{aligned}
B\left(f^{n_{i}-n_{i+1}}, f^{n_{i+1}+1}(T)\right. & \left., f^{n_{i+1}+1}(J)\right) \\
& \geqslant \exp \left\{-C\left(S\left(n_{i+1}, n_{i}, T\right)\right)^{\eta} \sum_{k=1}^{n_{i}-n_{i+1}}\left|f^{k+n_{i+1}}(T)\right|\right\} \\
& >\exp \left\{-C\left(S\left(0, \hat{n}(i), U_{i}^{s(i)}\right)\right)^{\eta} C_{\text {all }}\right\} \\
& >\exp \left\{-C\left(\gamma S\left(0, \hat{n}(i-1), U_{i-1}^{s(i-1)}\right)\right)^{\eta} C_{\text {all }}\right\} .
\end{aligned}
$$

If we are not in the infinite cascade case then the sums for $F_{i}, F_{i+1}, \ldots$ can be broken into blocks consisting of a cascade; possibly followed by an exceptional case; followed by one or more well bounded cases. So suppose that $F_{i}$ is well bounded, $F_{i}, F_{i+1}, \ldots, F_{i+m-1}$ have central returns, $F_{i+m}$ has a non-central return and $F_{i+m+1}$ is an exceptional case. So note that, in particular, $F_{i+m+2}$ must be well bounded. Then,

$$
\begin{array}{r}
S\left(0, \hat{n}(i+m+3), U_{i+m+3}^{s(i+m+3)}\right)<\gamma S\left(0, \hat{n}(i+m+2), U_{i+m+2}^{s(i+m+2)}\right), \ldots \\
\ldots, \gamma S\left(0, \hat{n}(i+1), U_{i+1}^{s(i+1)}\right)<\gamma^{2} S\left(0, \hat{n}(i), U_{i}^{s(i)}\right)
\end{array}
$$

Therefore, we have

$$
\begin{aligned}
B\left(f^{n}, T, J\right) & >\exp \left\{-C \sum_{k=0}^{n-1}\left|f^{k}(T)\right|^{1+\eta}\right\} \\
& >\exp \left\{-C C_{\text {all }}\left(S\left(0, \hat{n}(0), U_{0}^{s(0)}\right)\right)^{\eta} \sum_{k=0}^{\infty} \gamma^{k \eta}\right\} \\
& >\exp \left\{-C C_{\text {all }} \frac{\left(\sigma^{\prime}\left(\left|I_{0}\right|\right)\right)^{\eta}}{1-\gamma^{\eta}}\right\} .
\end{aligned}
$$

Hence it is easy to see that for any $0<K<1$, if $I_{0}$ is the central domain of a first return map to some $I_{-1}, I_{0}$ is sufficiently small and $F_{-1}$ is non-central, then we may bound $B\left(f^{n}, T, J\right)$ below by $K$.

Note that we can always start with a well-bounded case when we don't have an infinite cascade. We simply induce on a nice interval finitely many times until we obtain a non-central return and thus obtain a suitable $I_{-1}$. We consider the infinite cascade case in the next section.

The second part of Theorem 1.2, concerning $A\left(f^{n}, T, J\right)$, is proved in the same way.

## 7. Infinite cascade case

Here we consider the case where we have some $I_{0}$ such that $F_{i}$ are central for $i=0,1, \ldots$. In this case we will find that $\frac{\left|I_{i+1}\right|}{\left|I_{i}\right|}$ gets very close to 1 . See Figure 5 for an example of such a map. In particular, $I_{i}$ will not shrink down to a point (the critical point c) as $i$ increases so we can't use the method above to bound sums of intervals which land very close to $c$. The principal tool here is an extension given by a result of [K2]. We will not supply all the details of our proof of Theorem 1.2 in this case since the techniques are mostly the same as applied in the previous sections. We start by letting $I_{0}$ be any nice interval about $c$. We assume that we have some infinite cascade. This means that for a nice interval $I_{0} \ni c, F_{i}$ is central (and high) for all $i$, where $F_{i}$ is defined in the usual way. The main idea here is that we can still find good bounds on some interval $I_{0,0}$ and then apply the methods of Section 4 to it. Then we need to find another interval $I_{1,0}$ around $c$ which is smaller than all $I_{0, i}$, also has good bounds and is uniformly smaller than $I_{0,0}$. In such a way, we obtain a sequence of intervals $I_{i, 0}$ which can each be treated as in the high cascade case above, and which shrink uniformly to the critical point. Clearly $F_{i, j}$ will always be central and high for all $i, j \geqslant 0$.
Proposition 7.1. For $f \in N F^{2}$, and $\xi>0$ there exists some $C_{\text {inf }}>0$ such that for any small $I_{0,0}$ defined as above, $T \subset I_{0,0}$ implies

$$
\sum_{k=0}^{n-1}\left|f^{k}(T)\right|^{1+\xi}<C_{i n f}
$$



Figure 5. An infinite cascade.

Clearly this completes the proof of Theorem 1.2 in this case.

Proof. We will prove this with a series of lemmas.
For all $i$ the central branch of $F_{i}$ has two fixed points, $q_{0}$ and $p_{0}$ to the left and right of $c$ respectively (as usual, we assume that $F_{i}(c)$ is a maximum for $\left.\left.F_{i}\right|_{I_{i+1}}\right)$. We let $q_{0}^{\prime}$ be the point in $I_{i+1}$ not equal to $q_{0}$ which maps by $F_{i}$ to $q_{0}$. We define $p_{0}^{\prime}$ similarly. We define $I_{0,0}$ to be $\left(p_{0}^{\prime}, p_{0}\right)$. Let $F_{0,0}: \bigcup_{j} I_{0,0}^{j} \rightarrow I_{0,0}$ be the first return map to $I_{0,0}$ (where $I_{0,0}^{0}$ is the central domain). We have the following lemma.

Lemma 7.2. There exists some $\hat{\chi}>0$ depending only on $f$ such that $I_{0,0}$ is a $\hat{\chi}$-scaled neighbourhood of every domain $I_{0,0}^{j}$ which has $\partial I_{0,0}^{j} \cap \partial I_{0,0}=\emptyset$.

Proof. Clearly, $I_{i}$ tends to $\left(q_{0}, q_{0}^{\prime}\right)$. So we denote $\left(q_{0}, q_{0}^{\prime}\right)$ by $I_{\infty}$. We will first show that $I_{\infty}$ is uniformly larger than $I_{0,0}$ and then show that all except two non-central domains of the first entry map to $I_{0,0}$ have an extension to $I_{\infty}$ and show what this means for $I_{0,0}^{j}$. These two domains are the ones with either $p_{0}$ or $p_{0}^{\prime}$ in the closure.

In a similar manner to the exceptional case, we will find an upper bound for $\left|D F_{i}\right|_{I_{i+1}}$. This will allow us to get good bounds for the first return map to $I_{0,0}$

For large $i$, the ratio $I_{i}$ has $\frac{\left|I_{i+1}\right|}{\left|I_{i}\right|}$ close to 1 . The following lemma, an adaptation of Lemma 7.2 of $[\mathbf{K 2}]$, allows us to bound $\left|D F_{i}\right|_{I_{i+1}}$.
Lemma 7.3. If $f \in N F^{2}$ then there exist constants $0<\tau_{2}<1$ and $\tau_{3}>$ 0 with the following property. If $T$ is any sufficiently small nice interval around the critical point, $R_{T}$ is the first entry map to $T$ and its central domain $J$ is sufficiently big, i.e. $\frac{|J|}{|T|}>\tau_{2}$, then there is an interval $W$ which is a $\tau_{3}$-scaled neighbourhood of the interval $T$ such that if $c \in R_{T}(J)$ then the range of any branch of $R_{T}: V \rightarrow T$ can be extended to $W$ provided that $V$ is not $J$.

This lemma is only needed as a $C^{3}$ result in [K2], but it easily extends to our $C^{2}$ case.

It is straightforward to see that the above lemma is sufficient to prove a version of Lemma 5.5 in our case. That is, for large $i$, there exists some $\hat{C}^{\prime}$ such that $\left|D F_{i}\right|_{I_{i+1}}<\hat{C}^{\prime}$. This implies that there exists some $0<\theta<1$ depending only on $f$ such that $\left|I_{0,0}\right|<\theta\left|I_{\infty}\right|$ and, equivalently, some $\delta>0$ such that $I_{\infty}$ is a $\delta$-scaled neighbourhood of $I_{0,0}$.

Now, for the moment we let $F_{0,0}$ also denote the first entry map and $\bigcup_{j} I_{0,0}^{j}$ also include the first entry domains. We will show that many of the branches have an extension to a uniformly larger domain. Suppose that there exists a domain $I_{0,0}^{j}$ with $\overline{I_{0,0}} \cap \overline{I_{0,0}^{j}}=\emptyset$ such that $F_{0,0}: I_{0,0}^{j} \rightarrow I_{0,0}$ does not have an extension to $I_{\infty}$. That is, supposing $\left.F_{0,0}\right|_{I_{0,0}^{j}}=\left.f^{n(j)}\right|_{I_{0,0}^{j}}$, there is no interval $V \supset I_{0,0}^{j}$ such that $f^{n(j)}: V \rightarrow I_{\infty}$ is a diffeomorphism. Let $0 \leqslant k \leqslant n(j)-1$ be maximal such that $f^{n(j)-k}: f^{k}\left(I_{0,0}^{j}\right) \rightarrow I_{0,0}$ has no extension to $I_{\infty}$. Clearly if $I_{0,0}$ is small $f: f^{n(j)-1}\left(I_{0,0}^{j}\right) \rightarrow I_{0,0}$ always has an extension, so $k<n(j)-1$. Then there exists some interval $W \supset f^{k+1}\left(I_{0,0}^{j}\right)$ such that $f^{n(j)-k-1}: W \rightarrow I_{\infty}$ is a diffeomorphism and the element $W^{\prime}$ of $f^{-1}(V)$ containing $f^{k}\left(I_{0,0}^{j}\right)$ contains $c$.

Since $I_{\infty}$ is a nice interval, $W^{\prime} \subset I_{\infty}$. We also know that $f^{k}\left(I_{0,0}^{j}\right) \subset I_{\infty} \backslash I_{0,0}$. Therefore $W^{\prime}$ contains either $p_{0}$ or $p_{0}^{\prime}$. But then either $f^{n(j)-k-1}\left(p_{0}\right)$ or $f^{n(j)-k-1}\left(p_{0}^{\prime}\right)$ is contained in $I_{\infty} \backslash I_{0,0}$ which is not possible.

Consider $I_{0,0}^{j}$ for some $j \neq 0$ where $I_{0,0}^{j} \subset I_{0,0}$ is a domain of the first return map. We will show that this domain is uniformly deep inside $I_{0,0}$. There exists some $V \supset f\left(I_{0,0}^{j}\right)$, where $f^{n(j)}: V \rightarrow I_{\infty}$ is a diffeomorphism and $V$ is a $\tilde{\delta}$-scaled neighbourhood of $f\left(I_{0,0}^{j}\right)$. Let $V^{\prime}$ be the maximal interval around $I_{0,0}^{j}$ such that $f\left(V^{\prime}\right)=V$. We show that $V^{\prime} \subset I_{0,0}$. Let $V(f(c))$ denote the maximal interval around $f(c)$ which pulls back by $f^{-1}$ to $I_{0,0}$. If $V$ is not contained in $V(f(c))$ then either $p_{0}$ or $p_{0}^{\prime}$ is contained in $V^{\prime}$. Thus, $f^{n(j)}\left(p_{0}\right)$ or $f^{n(j)}\left(p_{0}^{\prime}\right)$ lies in $I_{\infty} \backslash I_{0,0}$, a contradiction. So $V^{\prime} \subset I_{0,0}$ and $I_{0,0}$
is a $\delta^{\prime}$-scaled neighbourhood of $I_{0,0}^{j}$ where $\delta^{\prime}=\min \left(\tilde{\delta}, \frac{1}{2}\right)$. The case of the central branch follows in the usual manner.

So we are in a type of high cascade case for $F_{0,0}$. Note that the branches with $p_{0}$ or $p_{0}^{\prime}$ in their closure can be dealt with in the same way as the domains $V^{1}, V^{2}$ were dealt with in the exceptional case.

We may assume that $F_{0,0}$ has an infinite cascade and is high too. Let $F_{0,1}$ be the first return map to $I_{0,0}$ and so on, so we obtain $I_{0, i}$. We sum for $F_{0,0}, F_{0,1}, \ldots$ as in the high cascade case. We let $q_{1}, q_{1}^{\prime}, p_{1}, p_{1}^{\prime}$ be defined as above for the fixed points of $\left.F_{0,0}\right|_{I_{0,1}}$. We let $I_{0, \infty}$ denote $\left(q_{1}, q_{1}^{\prime}\right)$. We may apply the same ideas as above to find some new interval $I_{1,0}:=\left(p_{1}, p_{1}^{\prime}\right)$ which has $\left|I_{1,0}\right|<\theta\left|I_{0, \infty}\right|$. We may define $I_{i, j}$ for $i \geqslant 2$, and $0 \leqslant j \leqslant \infty$ in a similar way.

Let $f^{N_{i}}(T)$ be the last iterate of $T$ which lies inside $I_{i, 0}$. Let $N_{i}^{\prime}$ be the maximal integer $N_{i} \geqslant N_{i}^{\prime}>N_{i+1}$ such that $f^{N_{i}^{\prime}}(T)$ is not in $I_{i, 0} \backslash I_{i, \infty}$. Then these arguments prove the following lemma.

Lemma 7.4. There exists some $C>0$ such that

$$
\sum_{k=1}^{N_{i}-N_{i}^{\prime}}\left|f^{k+N_{i}^{\prime}}(T)\right|<C \hat{\sigma}_{i}
$$

where $\hat{\sigma}_{i}$ is defined as follows. Let $\sigma_{i}:=\sup _{V \in \operatorname{dom} F_{i, 0}} \sum_{j=1}^{n(V)}\left|f^{j}(V)\right|$ (and $n(V)$ is defined as $k$ where $\left.\left.F_{i, 0}\right|_{V}=f^{k}\right)$. Let $\hat{V} \subset I_{i, 0} \backslash I_{i, 1}$ be an interval such that $f^{\hat{n}}(\hat{V})$ is one of the connected components of $I_{i, 0} \backslash I_{i, 1}$ and $f^{j}(\hat{V})$ is disjoint from both $I_{i, 0} \backslash I_{i, 1}$ and $I_{i+1,0}$ for $0<j<\hat{n}(\hat{V})$. Then $\hat{\sigma}_{i}$ is the supremum of all such sums $\sum_{j=1}^{\hat{n}(\hat{V})}\left|f^{j}(\hat{V})\right|$ and $\sigma_{i}$.

Now we consider $\sum_{k=1}^{N_{i}^{\prime}-N_{i+1}}\left|f^{k+N_{i+1}}(T)\right|$. If none of these intervals contain $p_{i}, q_{i}$ then we are in $I_{i, \infty} \backslash I_{i+1,0}$. By the Minimum principle, $\left|D F_{i}\right|_{I_{i, \infty} \backslash I_{i+1,0}}$ is uniformly greater than 1 . So we can easily bound our sum. If none of our intervals contains $p_{i}$, but some $f^{k+N_{i+1}}(T)$ contains $q_{0}, q_{0}^{\prime}$ we can split $f^{k+N_{i+1}}(T)$ at $q_{0}$ or $q_{0}^{\prime}$ into two intervals. It is easy to see that there is some $C>0$ such that $\sum_{k=1}^{N_{i}^{\prime}-N_{i+1}}\left|f^{k+N_{i+1}}(T)\right|<C \hat{\sigma}_{i}$. If $p_{0}, p_{0}^{\prime}$ is contained in some $\left|f^{k+N_{i+1}}(T)\right|$ then we must split the interval at $p_{0}$ or $p_{0}^{\prime}$. Note that we may have to split the interval $\left|f^{k+N_{i+1}}(T)\right|$ at arbitrarily many $p_{i}, q_{i}^{\prime}$ or $p_{i}^{\prime}, q_{i}$. Therefore, $\sum_{k=1}^{N_{i}^{\prime}-N_{i+1}}\left|f^{k+N_{i+1}}(T)\right|^{1+\xi}<C \sum_{k=i}^{\infty}(k-$ i) $S\left(N_{k}, \hat{n}(k), T\right)^{\xi} \hat{\sigma}_{k}$ where $S$ and $\hat{n}$ are defined analogously to Section 6. As before there is some constant $0<\theta^{\prime}<1$, here depending on $\theta$ rather than $\gamma$ such that $\theta^{\prime}$ governs the decay of $S\left(N_{i}, \hat{n}(i), T\right)$. Hence, we can put this
estimate together with Lemma 7.4 to get

$$
\sum_{k=1}^{N_{i}-N_{i+1}}\left|f^{k+N_{i+1}}(T)\right|^{1+\xi}<C S\left(N_{i}, \hat{n}(i), T\right)^{\xi} \sum_{k=0}^{\infty} k \theta^{\prime k \xi}
$$

Similarly to before, we can conclude that there exists some $C_{i n f}>0$ such that

$$
\sum_{k=0}^{n}\left|f^{k}(T)\right|^{1+\xi}<C_{i n f}
$$

## Appendix A. Proof of the Yoccoz Lemma

We recall the lemma.
Lemma 4.3 Suppose that $f \in N F^{2}$. Then for all $\delta, \delta^{\prime}>0$ there exists $C>0$ such that if $I_{0}$ is a nice interval such that
(1) $I_{0}$ is a $\delta$-scaled neighbourhood of $I_{1}$;
(2) $F_{i}$ is low and central for $i=0, \ldots, m$;
(3) there is some $0<i<m$ with $\frac{\left|I_{i}\right|}{\left|I_{i+1}\right|}<1+\delta^{\prime}$,
then for $1 \leqslant k<m$,

$$
\frac{1}{C} \frac{1}{\min (k, m-k)^{2}}<\frac{\left|I_{i+k-1} \backslash I_{i+k}\right|}{\left|I_{i}\right|}<\frac{C}{\min (k, m-k)^{2}} .
$$

For similar statements see $[\mathbf{F M}]$ and $[\mathbf{S h} 2]$.
Proof. We first point out the following claim.
Claim 1. For $f$ as in the lemma, there exists some $C\left(f, \delta, \delta^{\prime}\right)>0$ such that

$$
\frac{\left|I_{m}\right|}{\left|I_{0}\right|}>C\left(f, \delta, \delta^{\prime}\right)
$$

This is proved in Section 5 of [Sh2]. One consequence of this is that $\frac{\left|I_{m} \backslash I_{m+1}\right|}{\left|I_{0}\right|}$ is uniformly bounded below. This is one of the assumptions in the statement of the Yoccoz Lemma in $[\mathbf{F M}]$.

Our proof now involves using a result of $[\mathbf{S T}]$, the bound $\delta$ and the small size of $I_{0}$, to find a nearby map in the Epstein class. The structure of such maps, particularly at parabolic fixed points, along with some new coordinates, give us estimates for $\frac{\left|I_{i+k-1} \backslash I_{i+k}\right|}{\left|I_{i}\right|}$.
We suppose that $s>0$ is such that $\left.F_{0}\right|_{I_{1}}=\left.f^{s}\right|_{I_{1}}$. We observe that $f^{s-1}$ has uniformly bounded distortion depending on $\delta$. We will denote $\left.F_{0}\right|_{I_{1}}$ by
$F$. Letting $\psi:\left[a_{m}, a_{1}\right] \rightarrow[0,1]$ be an affine diffeomorphism we will work with the map $\psi \circ F \circ \psi^{-1}$. For the rest of the appendix we will abuse the notation and denote this map by $F$ too.

Previously we assumed that $\left.F\right|_{I_{1}}$ had a maximum at $c$. It will be convenient to suppose now for this section that $c$ is a minimum for $\left.F\right|_{I_{1}}$. Also we let $I_{i}=\left(a_{i}^{\prime}, a_{i}\right)$. So in particular, $F\left(a_{i+1}\right)=a_{i}$. We firstly define a point which allows us to partition $\left[a_{m}, a_{1}\right]$ in another way.

Let $x_{0} \in\left[a_{m}, a_{1}\right]$ be defined so that $\left|F\left(x_{0}\right)-x_{0}\right|=\min _{a_{m} \leqslant x \leqslant a_{0}}|F(x)-x|$. It is easy to show that $D F\left(x_{0}\right)=1$. We will suppose throughout that $\left|F\left(x_{0}\right)-x_{0}\right|$ shrinks to zero as $\left|I_{0}\right| \rightarrow 0$ : otherwise the proof is much simpler. We can estimate the shape of $F$ near $x_{0}$ using the following definition and lemma.

Let $\kappa>0$. We say that the real analytic map $f:[0,1] \rightarrow[0,1]$ is in the Epstein class $\mathcal{E}_{\kappa}$ if $f(x)=\varphi Q \psi$ where $Q$ is the quadratic map $Q(z)=z^{2}$, $\psi$ is an affine map and $\varphi:[0,1] \rightarrow[0,1]$ is a diffeomorphism whose inverse has a holomorphic extension which is univalent in the domain $\mathbb{C}_{(-\kappa, 1+\kappa)}:=$ $\mathbb{C} \backslash((-\infty,-\kappa] \cup[1+\kappa, \infty))$. For more details on maps in this class see [MS]. The following lemma is proved in [ST].

Lemma A.1. Let $f \in N F^{2}$. Suppose that $I$ is a nice interval around $c$ and $J$ is a first entry domain which is disjoint from $I$ and with entry time $s$. Suppose that $\delta>0$ is some constant such that there exists some $\hat{J} \supset J$ such that $f^{s}: \hat{J} \rightarrow I^{\prime}$ is a diffeomorphism where $I^{\prime}$ is a $\kappa$-scaled neighbourhood of $I$ and $\sum\left|f^{j}(\hat{J})\right| \leqslant 1$. Let $\tau_{0}: J \rightarrow[0,1]$ and $\tau_{s}: I \rightarrow[0,1]$ be affine diffeomorphisms. Then for all $\epsilon>0$ there exists $\delta>0$ such that $|I|<\delta$ implies that there exists some function $G: I \rightarrow I$ in the Epstein class $\mathcal{E}_{\frac{\kappa}{2}}$ such that $\left\|\tau_{s} \circ f^{s} \circ \tau_{0}^{-1}-G\right\|_{C^{2}}<\epsilon$.

We use this to prove the following claim.
Claim 2. There exists some $0<A<B$ such that for $I_{0}$ sufficiently small

$$
F\left(x_{0}\right)+\left(x-x_{0}\right)+A\left(x-x_{0}\right)^{2} \leqslant F(x) \leqslant F\left(x_{0}\right)+\left(x-x_{0}\right)+B\left(x-x_{0}\right)^{2} .
$$

Proof. We know that $f^{s}: I_{2} \rightarrow I_{1}$ has the following property. The map $f^{s-1}: f\left(I_{2}\right) \rightarrow I_{1}$ has an extension to $I_{0}$. Furthermore, since $I_{0}$ is a $\delta-$ scaled neighbourhood of $I_{1}$ we use Lemma A. 1 to obtain a map $G_{\infty}$ in the Epstein class which is $C^{2}$-close to $f^{s}$.

In fact we can choose different starting intervals $I_{n}$ with the same real bounds which are smaller and smaller and which are then rescaled to maps $F_{n}$ which map from the unit interval to itself. For each such map we obtain the nearby $\operatorname{map} G_{n}$ in the Epstein class where $\left\|F_{n}-G_{n}\right\|_{C^{2}} \rightarrow 0$ as $n \rightarrow \infty$.

For $F_{n}$ we let $x_{0}^{n}$ denote a point which is equivalent to $x_{0}$ for $F$. Since we assume that $\left|F_{n}\left(x_{0}^{n}\right)-x_{0}^{n}\right|$ goes to zero, our limit map $G_{\infty}$ has a parabolic fixed point $x_{0}^{\infty}$. Also $D^{2} G_{\infty}\left(x_{0}^{\infty}\right)>0$. Thus, there exist $0<A<B$ depending only on $f$ such that for all $x \in[0,1]$ we have
$G_{\infty}\left(x_{0}^{\infty}\right)+\left(x-x_{0}^{\infty}\right)+A\left(x-x_{0}^{\infty}\right)^{2} \leqslant G_{\infty}(x) \leqslant G_{\infty}\left(x_{0}^{\infty}\right)+\left(x-x_{0}^{\infty}\right)+B\left(x-x_{0}^{\infty}\right)^{2}$.
Clearly, for large $n$, we have the same condition for $G_{n}$. Therefore, if we take $I_{0}$ small enough, we may assume that it holds for $F$ too.

We denote $\epsilon:=F\left(x_{0}\right)-x_{0}$. Then we have

$$
\epsilon+A\left(x-x_{0}\right)^{2} \leqslant F(x)-x \leqslant \epsilon+B\left(x-x_{0}\right)^{2} .
$$

We suppose that $N$ is such that $x_{0} \in\left[a_{N}, a_{N+1}\right)$. Then for $0 \leqslant i \leqslant N-1$ we let $x_{i}:=F^{i}\left(x_{0}\right)$. We will use this equation to find estimates for $a_{j}-a_{j+1}$. Throughout we will let $C, C^{\prime}$ denote some constants depending only on $\delta, \delta^{\prime}$.

## Claim 3.

$$
N=O\left(\frac{1}{\sqrt{\epsilon}}\right)
$$

Proof. Let $N^{\prime}=\max \left\{1 \leqslant j \leqslant N-1: x_{j}-x_{0} \leqslant \sqrt{\epsilon}\right\}$. We will first show that $N^{\prime}$ satisfies the claim. For $j \leqslant N^{\prime}$, we have

$$
\epsilon \leqslant x_{j+1}-x_{0} \leqslant \epsilon(B+1)
$$

Therefore,

$$
N^{\prime} \epsilon \leqslant \sum_{j=0}^{N^{\prime}-1} x_{j+1}-x_{j} \leqslant N^{\prime} \epsilon(B+1)
$$

Since $\sum_{j=0}^{N^{\prime}-1} x_{j+1}-x_{j}=x_{N^{\prime}}-x_{0} \leqslant \sqrt{\epsilon}$ we have $N^{\prime} \leqslant \frac{1}{\sqrt{\epsilon}}$. Furthermore, $x_{j+1}-x_{0}>\sqrt{\epsilon}$ so $\epsilon\left(N^{\prime}(B+1)+1\right)>\sqrt{\epsilon}$ and $N^{\prime}>\frac{1}{(B+1) \sqrt{\epsilon}}-1$. I.e. $N^{\prime}=O\left(\frac{1}{\sqrt{\epsilon}}\right)$.
Next we find estimates for $N-N^{\prime}$. For $N^{\prime}<j \leqslant N$ we again consider the equation

$$
\epsilon+A\left(x_{j}-x_{0}\right)^{2} \leqslant x_{j+1}-x_{j} \leqslant \epsilon+B\left(x_{j}-x_{0}\right)^{2}
$$

But note that here $B\left(x_{j}-x_{0}\right)^{2}>\epsilon$ so we can write instead

$$
A\left(x_{j}-x_{0}\right)^{2} \leqslant x_{j+1}-x_{j} \leqslant 2 B\left(x_{j}-x_{0}\right)^{2}
$$

We make a change of coordinates. We let $y_{j}:=\frac{1}{x_{j}-x_{0}}$. Then we have

$$
y_{j}-y_{j+1}=\frac{x_{j+1}-x_{j}}{\left(x_{j}-x_{0}\right)\left(x_{j+1}-x_{0}\right)}
$$

By the above bounds we have

$$
\frac{A\left(x_{j}-x_{0}\right)}{x_{j+1}-x_{0}}<y_{j}-y_{j+1}<\frac{2 B\left(x_{j}-x_{0}\right)}{x_{j+1}-x_{0}}<2 B .
$$

Furthermore,
$y_{j}-y_{j+1}>\frac{A\left(x_{j}-x_{0}\right)}{\left(x_{j+1}-x_{j}\right)+\left(x_{j}-x_{0}\right)}>\frac{A\left(x_{j}-x_{0}\right)}{2 B\left(x_{j}-x_{0}\right)^{2}+\left(x_{j}-x_{0}\right)}>\frac{A}{2 B+1}$.
Observe that $x_{N} \in\left(a_{1}, a_{0}\right)$ and $\left|a_{0}-a_{1}\right|>\delta$. So since $\left|x_{N}-x_{N-1}\right|$ is approximately $\left|a_{0}-a_{1}\right|$ and since we fixed $\delta$, we know that $y_{N}=O(1)$. Also note that $y_{N^{\prime}}=O\left(\frac{1}{\sqrt{\epsilon}}\right)$ and so $y_{N^{\prime}}-y_{N}=O\left(\frac{1}{\sqrt{\epsilon}}\right)$. Summing we obtain

$$
\frac{C}{\sqrt{\epsilon}}<y_{N^{\prime}}-y_{N}=\sum_{j=N-1}^{N^{\prime}} y_{j}-y_{j+1}<2 B\left(N-N^{\prime}\right)
$$

and

$$
\frac{C^{\prime}}{\sqrt{\epsilon}}>y_{N^{\prime}}-y_{N}=\sum_{j=N-1}^{N^{\prime}} y_{j}-y_{j+1}>\frac{A\left(N-N^{\prime}\right)}{2 B+1}
$$

So $N-N^{\prime}=\left(\frac{1}{\sqrt{\epsilon}}\right)$ too. Adding this to the estimates for $N^{\prime}$ we prove the claim.

To prove the lemma, we will use Claims 1 and 3 together, along with bounded distortion, which means that $a_{j}-a_{j+1}$ is like $x_{N-j}-x_{N-j-1}$.

Firstly we will use the above coordinate change again. For $j>N^{\prime}$ we have

$$
y_{j}>y_{j}-y_{N}=\sum_{j=N-1}^{j} y_{i}-y_{i+1}>\frac{A(N-j)}{2 B+1}
$$

and so $\frac{1}{x_{j}-x_{0}}>\frac{A(N-j)}{2 B+1}$ and $x_{j+1}-x_{j}<2 B\left(\frac{2 B+1}{A(N-j)}\right)^{2}$.
We have proved that if $0 \leqslant j \leqslant N^{\prime}$ then

$$
\begin{equation*}
\epsilon<x_{j+1}-x_{j}<C^{\prime} \epsilon \tag{5}
\end{equation*}
$$

and if $N^{\prime}<j \leqslant N$ then

$$
\begin{equation*}
\epsilon<x_{j+1}-x_{j}<\frac{C^{\prime}}{(N-j)^{2}} \tag{6}
\end{equation*}
$$

Similarly we can define $x_{j}=F^{j}\left(x_{0}\right)$ for negative $j$ where $0 \leqslant|j|<m-N$. Now we will show that Claim 3 follows for this situation too and we get equivalents to (5) and (6). We define some $M^{\prime}$ analogously to the definition for $N^{\prime}$ and so if $|j| \leqslant M^{\prime}$ then

$$
\epsilon<x_{j+1}-x_{j}<C^{\prime} \epsilon .
$$

And if $M^{\prime}<|j| \leqslant m-N$ then

$$
\frac{C}{(m-N+j)^{2}}<x_{j+1}-x_{j}<\frac{C^{\prime}}{(m-N+j)^{2}} .
$$

(In the step of the proof where estimates on $y_{N-m}$ are required, we use Claim 1 to give $\left|a_{m-1}-a_{m}\right|$ uniformly bounded below and the fact that
$\left|x_{-m-1}-x_{m}\right|$ is approximately $\left|a_{m-1}-a_{m}\right|$.) Note also that we can show that $m-M^{\prime}=O\left(\frac{1}{\sqrt{\epsilon}}\right)$.

Observe that $a_{j}-a_{j+1}$ is essentially the same as $x_{N-j}-x_{N-j-1}$. So if $N \geqslant j \geqslant N-N^{\prime}$, we have

$$
C \epsilon<a_{j}-a_{j+1}<C^{\prime} \epsilon
$$

Observe that $\frac{1}{N-N^{\prime}} \geqslant \frac{1}{j} \geqslant \frac{1}{N}$. Since $\epsilon\left(\frac{1}{N^{2}}\right)$ and $\epsilon\left(\frac{1}{\left(N-N^{\prime}\right)^{2}}\right)$ this implies that we have

$$
\frac{C}{j^{2}}<a_{j}-a_{j+1}<\frac{C^{\prime}}{j^{2}}
$$

Now if $N-N^{\prime} \geqslant j \geqslant O(1)$ then clearly we have $a_{j}-a_{j+1}<\frac{C^{\prime}}{j^{2}}$. Also,

$$
\begin{aligned}
x_{N-j}-x_{N-j-1} & >A\left(x_{N-j-1}-x_{0}\right)^{2}=A\left(\sum_{k=j-1}^{N-1} x_{N-k}-x_{N-k-1}\right)^{2} \\
& \geqslant A\left(\sum_{k=1}^{N^{\prime}} x_{k}-x_{k-1}\right)^{2} \geqslant A\left(N^{\prime} \sqrt{\epsilon}\right)^{2}
\end{aligned}
$$

Now since $\sqrt{\epsilon}=O\left(\frac{1}{N^{\prime}}\right)$, we have $x_{N-j}-x_{N-j-1} \gtrsim 1$. Thus

$$
\frac{C}{j^{2}}<a_{j}-a_{j+1}<\frac{C^{\prime}}{j^{2}}
$$

If $N \leqslant j \leqslant m-M^{\prime}$ then again we have

$$
C \epsilon<a_{j}-a_{j+1}<C^{\prime} \epsilon
$$

Note that we also have $m-N \geqslant m-j \geqslant m-M^{\prime}$. Since $m-N, m-M^{\prime}=$ $\left(\frac{1}{\sqrt{\epsilon}}\right)$ we have

$$
\frac{C}{(m-j)^{2}}<a_{j}-a_{j+1}<\frac{C^{\prime}}{(m-j)^{2}}
$$

If $m-M^{\prime} \leqslant j \leqslant m-1$ we have

$$
\frac{C}{(m-j)^{2}}<a_{j}-a_{j+1}<\frac{C^{\prime}}{(m-j)^{2}}
$$

where the lower bound follows as above.
To conclude, if $1 \leqslant j \leqslant N$ then we have some constant $C$ such that $j \leqslant$ $C(m-j)$ and $a_{j}-a_{j+1} \asymp \frac{1}{j^{2}}$. If $N \leqslant j \leqslant m-1$ then we have some constant $C^{\prime}$ such that $m-j \leqslant C^{\prime} j$ and $a_{j}-a_{j+1} \asymp \frac{1}{(m-j)^{2}}$. So in either case we have

$$
a_{j}-a_{j+1} \asymp \frac{1}{(\min (j, m-j))^{2}}
$$

as required.

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