# PERIODS, LEFSCHETZ NUMBERS AND ENTROPY FOR A CLASS OF MAPS ON A BOUQUET OF CIRCLES 

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#### Abstract

We consider some smooth maps on a bouquet of circles. For these maps we can compute the number of fixed points, the existence of periodic points and an exact formula for topological entropy. We use Lefschetz fixed point theory and actions of our maps on both the fundamental group and the first homology group.


## 1. Introduction and statement of main Results

We will consider a particular class of maps on a bouquet of circles. We can characterise the periods of periodic orbits, Lefschetz numbers and entropy for this class.

We first recall the concept of Lefschetz number of period $n$. Let $M$ be a compact ANR of dimension $n$, see $[3,4]$. A continuous map $f: M \rightarrow M$ induces an endomorphism $f_{* k}: H_{k}(M, \mathbb{Q}) \rightarrow H_{k}(M, \mathbb{Q})$ for $k=0,1, \ldots, n$ on the rational homology of $M$. For a linear operator $A$, we let $\operatorname{Tr}(A)$ denote the trace of $A$. The Lefschetz number of $f$ is defined by

$$
L(f)=\sum_{k=0}^{n}(-1)^{k} \operatorname{Tr}\left(f_{* k}\right)
$$

Since $f_{* k}$ are integral matrices, $L(f)$ is an integer. By the well known Lefschetz Fixed Point Theorem, if $L(f) \neq 0$ then $f$ has a fixed point (see, for instance, [3]). We can consider $L\left(f^{m}\right)$ too: $L\left(f^{m}\right) \neq 0$ implies that $f^{m}$ has a fixed point. However, a fixed point of $f^{m}$ is not necessarily a periodic point of period $m$. Therefore, a function for detecting the presence of periodic points of a given period was given in [9]. This is the Lefschetz number of period $m$, defined as

$$
l\left(f^{m}\right)=\sum_{r \mid m} \mu(r) L\left(f^{\frac{m}{r}}\right)
$$

where $\sum_{r \mid m}$ denotes the sum over all positive divisors of $m$, and $\mu$ is the Moebius function defined as

$$
\mu(m)= \begin{cases}1 & \text { if } m=1 \\ 0 & \text { if } k^{2} \mid m \text { for some } k \in \mathbb{N} \\ (-1)^{r} & \text { if } m=p_{1} \cdots p_{r} \text { for distinct prime factors }\end{cases}
$$

According to the Moebius Inversion Formula (MIF), see for example [16],

$$
L\left(f^{m}\right)=\sum_{r \mid m} l\left(f^{r}\right)
$$

Define $\operatorname{Fix}\left(f^{m}\right)$ to be the set of fixed points of $f^{m}$ for all $m \in \mathbb{N}$, and define $\operatorname{Per}_{m}(f)$ to be the set of periodic points of period $m$. Let $\operatorname{Per}(f)$ denote the set of periods of the periodic points of $f$.

We consider maps on graphs. In particular we consider bouquets of circles as follows. For more details on such maps see [11] and [12]. We consider a set $S_{1}, \ldots, S_{n}$ in the plane where for $1 \leq i \leq n, S_{i}$ is diffeomorphic to the unit circle. We call each $S_{i}$ a circle and suppose further that they are nested inside each other and are pairwise disjoint, except at a single point $b$ where they all touch. We call this set $G_{n}$ and call $b$ the branching point. See Figure 1 for a picture of some $G_{3}$. We give each circle the anticlockwise orientation. With our graph arranged in such a way, the orientation on every circle is easy to see. We say that any graph $G \subset \mathbb{R}^{n}$ which is homotopic to some $G_{n}$ is a bouquet of circles.


Figure 1. $G_{3}$
Let $x, y \in S_{i}$ where $x \neq y$. Then let $[x, y]$ denote the closed arc in $S_{i}$ which starts at $x$, proceeds anticlockwise, and ends at $y$. Furthermore, for $x \in S_{i}$ and $y \in S_{j}$, we consider the connected set $[x, y]:=[x, b] \cup[b, y]$ to be an arc. Also, we consider $\{x\}$ to be a degenerate arc. We can extend this definition to the open arc $(x, y)$ and the half open $\operatorname{arcs}(x, y]$ and $[x, y)$ in the natural way. Note that any arc is homotopic to a point.

Any continuous map $f: G_{n} \rightarrow G_{n}$ induces an action on $H_{1}\left(G_{n}, \mathbb{Q}\right)=\overbrace{\mathbb{Q} \oplus \cdots \oplus \mathbb{Q}}^{n}$, the first homology group. We denote this action by $f_{* 1}: H_{1}\left(G_{n}, \mathbb{Q}\right) \rightarrow H_{1}\left(G_{n}, \mathbb{Q}\right) . f_{* 1}$ can be represented by an $n \times n$ integral matrix $\left(m_{i j}\right)$ such that a generator $a_{j} \in H_{1}\left(G_{n}, \mathbb{Q}\right)$ maps by $f_{* 1}$ to the generator $a_{i}, m_{i j}$ times, taking into account orientation. See, for example, [18] for more details.

For a continuous map $f: M \rightarrow M$ on a compact ANR $M$, we define the minimal set of periods for $f$ to be the set

$$
\operatorname{MPer}(f)=\bigcap_{g \simeq f} \operatorname{Per}(g)
$$

where $\simeq$ denotes homotopy. In [11] and [12] the following was proved for continuous maps on $G_{n}$.

Theorem 1. Let $f: G_{n} \rightarrow G_{n}$ be a continuous map and let $f_{* 1}$ be the $n \times n$ integral matrix induced on the first homology group of $G_{n}$. Then the following statements hold.
(a) If there is some element of the diagonal of $f_{* 1}$ different from $-2,-1,0,1$, then $\operatorname{MPer}(f)=\mathbb{N}$.
(b) If all the elements of the diagonal of $f_{* 1}$ are $-2,-1,0$ or 1 , and at least one of them is -2 then $\operatorname{MPer}(f)=\mathbb{N}$ or $\mathbb{N} \backslash\{2\}$.

Any map $f: G_{n} \rightarrow G_{n}$ has a lift to a map $\tilde{f}:[0, n] \rightarrow[0, n]$ as follows. We identify the integers $0,1, \ldots, n$ with $b$ and identify $[i-1, i)$ with $S_{i}$. We assume that the lifting map $\pi:[0, n] \rightarrow G_{n}$ is continuous, is orientation preserving and is $C^{1}$ on each $x \in(i-1, i)$ for $1 \leq i \leq n$. Note that $\pi$ is an example of a covering map (see [18]).

We will consider the following class of maps, for which we can prove more. We let $f$ be a continuous map $f: G_{n} \rightarrow G_{n}$ which is (1) $C^{1}$ on $G_{n} \backslash\{b\} ;(2)$ for any $m \geq 1$, for $x \in \operatorname{Fix}\left(f^{m}\right) \backslash\{b\},\left|D f^{m}(x)\right|>1$; and (3) the sign of the derivative of the lift $\tilde{f}$, $\operatorname{sign}(D \tilde{f}(x))$ for $x \in(0, n) \backslash \mathbb{N}$ is constant. Any such map is monotone and we say that it is in $\mathcal{M}^{n}$. If, furthermore, $f^{m}(b) \neq b$ for all $m \geq 1$ then we say that $f$ is in $\mathcal{M}_{b}^{n}$. Note that any $f \in \mathcal{M}_{b}^{n}$ is either, orientation preserving on all of $G_{n}$, or orientation reversing on all of $G_{n}$.

Our first result on maps in this class is the following.
Theorem 2. Suppose that $f \in \mathcal{M}_{b}^{n}$. Then,
(a) for all $m \geq 1$, if $f^{m}$ is orientation preserving then $L\left(f^{m}\right)=-\# \operatorname{Fix}\left(f^{m}\right)$;
(b) for all $m \geq 1$, if $f^{m}$ is orientation reversing then $L\left(f^{m}\right)=\# \operatorname{Fix}\left(f^{m}\right)$;
(c) if $f$ is orientation preserving then $\left|l\left(f^{m}\right)\right|=\# \operatorname{Per}_{m}(f)$;
(d) if $f$ is orientation reversing and either $m$ is odd or $4 \mid m$, then $\left|l\left(f^{m}\right)\right|=\# \operatorname{Per}_{m}(f)$.

We next find formulae for the number of fixed points of maps in terms of the action on the fundamental group. We will find a class of maps $\mathcal{M}_{\#}^{n}$ which have an action on the fundamental group which corresponds well with maps in $\mathcal{M}^{n}$.

For each circle $S_{j}$ for $1 \leq j \leq n$ there exists a corresponding generator in $\Pi\left(G_{n}\right)$. We label this generator $a_{j}$. We may assume that these are all positively oriented (that is, each $a_{j}$ corresponds to a circle with anticlockwise orientation).

We say that a word $b_{1} \ldots b_{m}$ is allowed by $\mathcal{M}_{\#}^{n}$ if either all $b_{k} \in\left\{a_{1}, \ldots, a_{n}\right\}$ or all $b_{k} \in\left\{a_{1}^{-1}, \ldots, a_{n}^{-1}\right\}$. For a word $b_{1} \ldots b_{m}$ allowed by $\mathcal{M}_{\#}^{n}$, define

$$
\chi_{j}\left(b_{1} \ldots b_{m}\right)= \begin{cases}\#\left\{b_{k}=a_{j}: 1 \leq k \leq m\right\} & \text { if this set is not null, } \\ -\#\left\{b_{k}=a_{j}^{-1}: 1 \leq k \leq m\right\} & \text { if this set is not null, } \\ 0 & \text { otherwise. }\end{cases}
$$

Similarly we define

$$
\gamma_{j}\left(b_{1} \ldots b_{m}\right)= \begin{cases}\#\left\{b_{k}=a_{j}: 1<k<m\right\} & \text { if this set is not null, } \\ -\#\left\{b_{k}=a_{j}^{-1}: 1<k<m\right\} & \text { if this set is not null, } \\ 0 & \text { otherwise }\end{cases}
$$

Observe the difference between these two functions: $\chi_{j}$ counts the number of appearances of $a_{j}$ or $a_{j}^{-1}$ in $b_{1} \ldots b_{n}$, but $\gamma_{j}$ counts the number of appearances of $a_{j}$ or $a_{j}^{-1}$ in $b_{2} \ldots b_{n-1}$. So, for example $\chi_{j}\left(a_{j} a_{j+1} a_{j}\right)=2$, but $\gamma_{j}\left(a_{j} a_{j+1} a_{j}\right)=0$.

Now, for each $1 \leq j \leq n$, define $A_{j}$ to be the word $f_{\#}\left(a_{j}\right)$. We say that $f \in \mathcal{M}_{\#}^{n}$ if all $A_{j}$ are allowed by $\mathcal{M}_{\#}^{n}$. Note that $\mathcal{M}^{n} \subset \mathcal{M}_{\#}^{n}$. We define $d_{i j}:=\chi_{i}\left(A_{j}\right)$.

For $1 \leq k<\infty$, we say that the map $f \in \mathcal{M}^{n}$ is in $\mathcal{M}_{b, k}^{n}$ if $f^{k}(b)=b$, but there is no $1 \leq m<k$ such that $f^{m}(b) \neq b$. We say that $\mathcal{M}_{b, \infty}^{n}=\mathcal{M}_{b}^{n}$.

Proposition 3. If $f \in \mathcal{M}_{b, k}^{n}$ for some $1 \leq k \leq \infty$ then for any $m \notin k \mathbb{N}$, we have

$$
\# \operatorname{Fix}\left(f^{m}\right)=\left|1-\sum_{j=1}^{n} \chi_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|
$$

and if $k<\infty$, then for any $m \in k \mathbb{N}$ we have

$$
\# \operatorname{Fix}\left(f^{m}\right)=1+\left|\sum_{j=1}^{n} \gamma_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|
$$

Remark 4. For our maps the action on the fundamental group and that on the first homology group are very closely related. However, we see by the second part of this proposition that the fundamental group is particularly useful when studying fixed points of maps in $\mathcal{M}_{b, k}^{n}$ for $k<\infty$. In Theorem 2 we were not able to find an exact formula for the number of fixed points from the Lefschetz number for maps in this class. In fact, adding the above result to the formula for the Lefschetz number given by the action on the first homology group, it is possible to show that for such maps, for $m \in k \mathbb{N}$, $L\left(f^{m}\right) \leq \# \operatorname{Fix}\left(f^{m}\right) \leq 2 n-1+L\left(f^{m}\right)$.

Next we prove results on periods for maps in $\mathcal{M}_{b}^{n}$.
Proposition 5. For $f \in \mathcal{M}^{n}$, suppose that either (a) $\left|d_{j j}\right| \geq 2$ for some $1<j \leq n$; (b) $d_{11} \geq 2$; (c) $d_{11}<-2$; or (d) $f \in \mathcal{M}_{b, 1}^{n}$ and $d_{11}=-2$. Then $\operatorname{Per}(f)=\mathbb{N}$. Furthermore, if $(e) d_{11}=-2$ then $\operatorname{Per}(f) \supset \mathbb{N} \backslash\{2\}$.

This is essentially the same as Theorem 1 for maps in $\mathcal{M}^{n}$. But we prove it here for completeness. We can further characterise the set of periods in the following case.

Proposition 6. Suppose that $f \in \mathcal{M}_{b}^{n}$. Then we have the following.
(a) If there exist $1<i, j \leq n, i \neq j$ such that $\left|d_{i j}\right|,\left|d_{j i}\right| \geq 1$ and $\left|d_{i i}\right|+\left|d_{j j}\right| \geq 1$, then $\operatorname{Per}(f)=\mathbb{N}$.
(b) If there exists some $1<i \leq n$ such that $d_{i 1} \neq 0,-1$, then $\operatorname{Per}(f) \supset \mathbb{N} \backslash\{1\}$.
(c) If there exists some $1<i \leq n$ such that $d_{i 1}=-1$, then for all $m \geq 1$ either $m \in \operatorname{Per}(f)$ or $m+1 \in \operatorname{Per}(f)$.
Now suppose that $f \in \mathcal{M}_{b, 1}^{n}$. Then
(d) if there exist $1 \leq i, j \leq n, i \neq j$ such that $\left|d_{i j}\right|,\left|d_{j i}\right| \geq 1$ and $\left|d_{i i}\right|+\left|d_{j j}\right| \geq 1$, then $\operatorname{Per}(f)=\mathbb{N}$.

We next use the matrix $f_{* 1}$ to compute the entropy for maps on $G_{n}$. For some similar results on a different class of maps see the recent preprint [2]. We let the spectral radius of a linear map $L$ be equal to the largest modulus of the eigenvalues of this map. We denote this value by $\sigma(L)$. Let $h(f)$ denote the topological entropy of the map $f$, see Section 4 for details. Manning in [13] proved the following: a step towards proving the well-known entropy conjecture, proposed by Shub in [17].
Theorem 7. For any continuous map $f: M \rightarrow M$, for a compact differentiable manifold without boundary $M$, we have $h(f) \geq \log \sigma\left(f_{* 1}\right)$.

Following the arguments of [15] we can prove the following. Here, given an $n \times n$ $\operatorname{matrix} M$ we let $\|M\|:=\sum_{i, j}\left|m_{i j}\right|$.

Theorem 8. For a map $f \in \mathcal{M}^{n}$ we have (a) $h(f)=\lim _{m \rightarrow \infty} \frac{1}{m} \log \left\|f_{* 1}^{m}\right\|$; and (b) $h(f)=\log \sigma\left(f_{* 1}\right)$.

Given $f \in \mathcal{M}_{b}^{n}$, the map $f_{* 1}$ has eigenvalues $\lambda_{1}, \ldots, \lambda_{d}$ where the eigenvalues are in order of decreasing modulus $\left|\lambda_{1}\right| \geq \cdots \geq\left|\lambda_{d}\right|$ (when two eigenvalues have the same modulus, any choice of order suffices). Our final main result is as follows.

Proposition 9. For $f \in \mathcal{M}^{n}$ where the eigenvalues of $f_{* 1}$ have $\left|\lambda_{1}\right|>1$ and $\left|\lambda_{1}\right|>\left|\lambda_{2}\right|$, then there exists some $m_{0} \geq 1$ such that $m \geq m_{0}$ implies $m \in \operatorname{Per}(f)$.

Remark 10. It should be possible to extend these results to maps $f: G \rightarrow G$ for graphs $G$ which are homotopic to some $G_{n}$. We should also be able to extend some of the results to some classes of maps on some spaces which are homotopic to some $G_{n}$. For example, some class of maps on the disk punctured $n$ times (for maps on the twice punctured disk see [7]). However, it is difficult to characterise such maps.

In Section 2 we prove Theorem 2. In Section 3 we show that the action of maps on this class is well characterised by the action on the fundamental group and so prove Proposition 3. We then go on to prove Propositions 5 and 6. In Section 4 we prove Theorem 8. In Section 5 we prove Proposition 9. For examples of maps which we can apply our results to, see Section 6.

## 2. Applying Lefschetz numbers to a bouquet of circles

In this section we prove Theorem 2 and explain the problems associated with part (d) of the theorem.

First we recall that when $f: G_{n} \rightarrow G_{n}$ is $C^{1}$ and the fixed points of $f$ are isolated, we can express

$$
L(f)=\sum_{f(x)=x} \operatorname{ind}(f, x)
$$

where $\operatorname{ind}(f, x)$ is the index of $f$ at $x$. If $x \neq b$ then $\operatorname{ind}(f, x)=(-1)^{u_{+}(x)}$, where $u_{+}(x)=1$ whenever $D f(x)>1$ and $u_{+}(x)=0$ otherwise. For more details see [8] or [10]. There, the question of the index of $f$ at $b$ when $b$ is a fixed point is also discussed.

Proof of Theorem 2. The first two statements of the theorem are easy to see because,

$$
L(f)=\sum_{f(x)=x} \operatorname{ind}(f, x)=\sum_{f(x)=x}(-1)^{u_{+}(x)}
$$

where $u_{+}(x)$ is defined as above.
So $L(f)$ counts the number of fixed points, giving negative or positive sign if $f$ is orientation preserving or reversing, respectively. So we have proved (a) and (b).

Next we prove (c). Since $f$ is orientation preserving, the summands for $l\left(f^{m}\right)$ are all negative. Therefore, by the MIF,

$$
\sum_{r \mid m} \# \operatorname{Per}_{r}(f)=\left|L\left(f^{m}\right)\right|
$$

From the definition of $l(f)$, applying the MIF again we have $\left|l\left(f^{m}\right)\right|=\# \operatorname{Per}_{m}(f)$.
To prove (d), we first suppose that $m$ is odd. Then the summands for $l\left(f^{m}\right)$ are of the form $\mu(r) L\left(f^{\frac{m}{r}}\right)$ where $r \mid m$. Since $\frac{m}{r}$ cannot be even, $L\left(f^{\frac{m}{r}}\right)$ are either all negative or
all positive depending on whether $f$ is orientation preserving or reversing, respectively. Therefore, by the MIF,

$$
\sum_{r \mid m} \# \operatorname{Per}_{r}(f)=\left|L\left(f^{m}\right)\right|
$$

Again, a further application of the MIF gives $\left|l\left(f^{m}\right)\right|=\# \operatorname{Per}_{m}(f)$.
Now if $4 \mid m$ then let $n$ be such that $m=4 n$. Any summand for $l\left(f^{m}\right)$ is of one of the following forms.
(i) $\mu(r) L\left(f^{\frac{4 n}{r}}\right)$ where $r \mid 4 n$ and $r$ is odd (so $\left.r \mid n\right)$. Since $\frac{4 n}{r}$ is even, $L\left(f^{\frac{4 n}{r}}\right)$ is negative in any case.
(ii) $\mu(2 r) L\left(f^{\frac{2 n}{r}}\right.$ ) where $r \mid 2 n$ and $r$ is odd (so $\left.r \mid n\right)$. Since $\frac{2 n}{r}$ is even, $L\left(f^{\frac{2 n}{r}}\right.$ ) is negative in any case.
(iii) $\mu(4 r) L\left(f^{\frac{n}{r}}\right)$ where $r \mid n$. Since $\mu(4 r)=0$, this term is null.

Thus all of the terms $L\left(f^{\frac{m}{r}}\right)$ which contribute to $l\left(f^{m}\right)$ are negative and so, applying the MIF as above we see that

$$
\left|l\left(f^{m}\right)\right|=\# \operatorname{Per}_{m}(f)
$$

Remark 11. We cannot extend this method directly to maps with attracting periodic points, even if they are monotone. For example, we can create a monotone $C^{1}$ map which has every repelling fixed point followed by an attracting one. So we can have $L(f)=0$, where $f$ has arbitrarily many fixed points. (If $f$ is orientation reversing this does not make any difference for $L(f)$. But this presents a problem for $L\left(f^{2}\right)$.)

Remark 12. We explain why this result cannot be extended to $m$ where $2 \mid m$, but $4 \nmid m$ when $f$ is orientation reversing. Suppose that $m=2 p$ for some $p$ prime (we obtain similar problems if $m=2 p_{1} \ldots p_{r}$ with $p_{i}>2$ prime). Since $f$ is orientation reversing,

$$
\begin{aligned}
l\left(f^{2 p}\right)= & L\left(f^{2 p}\right)-L\left(f^{p}\right)-L\left(f^{2}\right)+L(f) \\
= & -\left[\# \operatorname{Per}_{2 p}(f)+\# \operatorname{Per}_{p}(f)+\# \operatorname{Per}_{2}(f)+\# \operatorname{Per}_{1}(f)\right] \\
& -\left[\# \operatorname{Per}_{p}(f)+\# \operatorname{Per}_{1}(f)\right]+\left[\# \operatorname{Per}_{2}(f)+\# \operatorname{Per}_{1}(f)\right]+\# \operatorname{Per}_{1}(f) \\
= & -\# \operatorname{Per}_{2 p}(f)-2 \# \operatorname{Per}_{p}(f)
\end{aligned}
$$

So, even when $l\left(f^{m}\right) \neq 0$, we cannot be so sure about the presence of periodic points of period $m$. This is seen in the following examples.

It is convenient to construct our examples on the level of homology where we only have information about $f_{* 1}: H_{1}\left(G_{n}, \mathbb{Q}\right) \rightarrow H_{1}\left(G_{n}, \mathbb{Q}\right)$.

Example 13. Consider the map $f \in \mathcal{M}^{1}$ which has action $f_{* 1}$ on $H_{1}\left(G_{1}, \mathbb{Q}\right)$ equal to multiplication by $m_{11}$ where $m_{11}=-2$. Then $f_{* 1}^{2}$ is multiplication by 4 . We calculate $L(f)=3, L\left(f^{2}\right)=-3$. So $l\left(f^{2}\right)=-6$. By Remark 12 we have

$$
l\left(f^{2}\right)=-\# \operatorname{Per}_{2}(f)-2 \# \operatorname{Per}_{1}(f)
$$

So we deduce that $\operatorname{Per}_{2}(f)=\emptyset$. Therefore, the Lefschetz number for periodic points does not always detect periodic points of even order when the original map is orientation reversing.

We can also construct further such examples for any $n \geq 2$ as follows. See Figure 2 for an example on $G_{3}$. Let $\left(m_{i j}\right)$ be the matrix representing the action of $f_{* 1}$ on
$H_{1}\left(G_{n}, \mathbb{Q}\right)$. Now suppose that $m_{11}=-2 ; m_{1 j}=-1$ for $1 \leq j \leq n$; and $m_{i j}=0$ for $0<i \leq n, 1 \leq j \leq n$. Here we obtain the same behaviour on $S_{1}$ as on $G_{1}$ above (note that there are no periodic points outside $S_{1}$ here). Therefore, we cannot be sure in such cases that $l\left(f^{m}\right) \neq 0$ implies that there are periodic points of period $m$.


Figure 2. Lift for a map $f \in \mathcal{M}_{b}^{3}$ where $l\left(f^{2}\right) \neq 0$, but $2 \notin \operatorname{Per}(f)$.

## 3. Finding periods from the action on the fundamental group

In fact, most of the information on periodic points for maps in $\mathcal{M}^{n}$ can be read from the action on the fundamental group $\Pi\left(G_{n}\right)$. We will see that there is a one to one correspondence between maps with a particular type of action on $\Pi\left(G_{n}\right)$ and homology classes of maps in $\mathcal{M}^{n}$. (As we will note later, this is not the case when we consider the action on first homology.)
3.1. Coding of $f$ on the fundamental group. If a word $b_{1} \ldots b_{m}$ is allowed by $\mathcal{M}_{\#}^{n}$ and has $\left(\left\{b_{1}\right\} \cup\left\{b_{m}\right\}\right) \cap\left(\left\{a_{1}\right\} \cup\left\{a_{1}^{-1}\right\}\right) \neq \emptyset$ then we say that $b_{1} \ldots b_{m}$ is allowed by $\mathcal{M}_{\# b}^{n}$. Note that a map with the action $f_{\#}: a_{j} \mapsto a_{j_{1}} \ldots a_{j_{n_{j}}}$ for $j_{k} \in\{1, \ldots, n\}$ which starts and finishes at the same point in the circle corresponding to $a_{j_{1}}$, is homotopic to a map with the action $f_{\#}: a_{j} \mapsto a_{j_{2}} \ldots a_{j_{n_{j}}} a_{j_{1}}$. We can argue analogously in the orientation reversing case. So for maps in $\mathcal{M}_{\#}^{n}$ with $f^{k}(b) \neq b$ for all $k \geq 1$, we may assume that $a_{j_{1}} \in\left\{a_{1}, a_{1}^{-1}\right\}$ for all $1 \leq j \leq n$. Observe that the action $a_{j} \mapsto a_{1} a_{j_{2}} \ldots a_{j_{n_{j}}}$ gives an orientation preserving map on $G_{n}$ which starts at $f(b)$, then covers the arc $[f(b), b]$; then covers in turn the circles $S_{j_{2}}, \ldots, S_{j_{n_{j}-1}}$ and $S_{j_{n_{j}}}$; finally it covers the arc $[b, f(b)]$.

Lemma 14. Suppose that $A_{1}, \ldots A_{n}$ are allowed by $\mathcal{M}_{\# b}^{n}$ and if $d_{j 1}=0$ for all $1<$ $j \leq n$ then $d_{11} \neq-1$. Then there exists a map $f \in \mathcal{M}^{n}$ with the action $f_{\#}: a_{j} \rightarrow A_{j}$ for $1 \leq j \leq n$. Furthermore, any $g \in \mathcal{M}_{\#}^{n}$ with the same action is homotopic to $f$.

Proof. We will find a piecewise linear lift map $\tilde{g}:[0, n] \rightarrow[0, n]$ with the required action and then show that $\tilde{f}:[0, n] \rightarrow[0, n]$, the lift of $f$ must be homotopic to $\tilde{g}$.

For an interval $J$, and a linear map $g: J \rightarrow \mathbb{R}$, let $|D g|_{J}=|D g(x)|$ for any $x \in J$. Given $1 \leq i \leq n$, we consider the word $A_{i}$. We let $\tilde{g}:[0, n] \rightarrow[0, n]$ be the piecewise linear map with $\tilde{g}(j)=\frac{1}{2}$ for $0 \leq j \leq n ;|D \tilde{g}|_{(j-1, j)}=\operatorname{sign}\left(\chi_{1}\left(a_{1}\right)\right) n_{j}$; first the map has $\tilde{g}(j-1)=\frac{1}{2}$; then it covers half of $[0,1]$ before covering the intervals $[i-1, i]$ given in $A_{j}$ in the order given by $A_{j}$; finally the map covers $\left[0, \frac{1}{2}\right]$ where $\tilde{g}(j)=\frac{1}{2}$.

For example if $f: G_{3} \rightarrow G_{3}$ and $f_{\#}: a_{1} \mapsto a_{1} a_{3} a_{1} a_{2} a_{2}$ then $\left.D \tilde{g}\right|_{[0,1]}=5$ and has $\tilde{g}\left(\left[0, \frac{1}{10}\right)\right)=\left[\frac{1}{2}, 1\right), \tilde{g}\left(\left[\frac{1}{10}, \frac{3}{10}\right)\right)=[2,3), \tilde{g}\left(\left[\frac{3}{10}, \frac{5}{10}\right)\right)=[0,1), \tilde{g}\left(\left[\frac{5}{10}, \frac{7}{10}\right)\right)=[1,2)$, $\tilde{g}\left(\left[\frac{7}{10}, \frac{9}{10}\right)\right)=[1,2)$, and $\tilde{g}\left(\left[\frac{9}{10}, 1\right)\right)=\left[0, \frac{1}{2}\right)$.

We now show that $\tilde{g}$ is homotopic to $\tilde{f}$. We first may assume that $\tilde{f}$ has been 'pulled tight'. That is, we choose a homotopy which results in a local homeomorphism, i.e. given any $0 \leq j \leq n$, for all $x \in(j-1, j)$ there exists a neighbourhood $U$ of $x$ such that $\left.\tilde{f}\right|_{U}$ is a homeomorphism. This means that the graph of $f$ has no null homotopic loops.

Suppose that $f$ is orientation preserving. For $1 \leq j \leq n$, let $I_{j 1}$ be the minimal interval in $[0,1]$ such that $\tilde{f}: I_{j 1} \rightarrow[\tilde{f}(0), 1]$ is a surjection. Let $\hat{I}_{j 1}$ be the equivalent interval for $\tilde{g}$. Since $\tilde{f}$ is assumed to be a local homeomorphism, $\tilde{f}_{I_{j 1}}$ is a homeomorphism. Since $\tilde{f}_{I_{j 1}}$ and $\tilde{g}_{\hat{I}_{j 1}}$ are both homeomorphisms on intervals with the same orientation then they are homotopic.

Now for any small enough interval $U$ adjacent and to the right of $I_{j 1}$ we claim that $\tilde{f}(U) \subset\left[j_{2}-1, j_{2}\right]$. If not then there is some $i \neq j_{2}$ such that $\tilde{f}(U) \subset[i-1, i]$. But since $\tilde{f}$ is a local homeomorphism, we can extend $U$ so that $\tilde{f}(U)=[i-1, i]$. But then $a_{j_{2}}=a_{i}$ which is a contradiction. As above, we can show that $\tilde{f}_{I_{j 2}}$ and $\tilde{g}_{\tilde{I}_{j 2}}$ are homotopic. We may continue this process up to $n_{j}$ to prove that $\tilde{f}$ and $\tilde{g}$ are homotopic.

Next we need to show that $\tilde{g}$ gives a map $g: G_{n} \rightarrow G_{n}$ which is in $\mathcal{M}^{n}$. We need to show that for any fixed point $x$ of $\tilde{g}^{m}$, we have $\left|D \tilde{g}^{m}(x)\right|>1$. We fix some $1 \leq j \leq n$ and consider $(j-1, j)$. We have the following cases.

Case 1: There exists some $i \neq 1$ such that $\left|d_{i j}\right| \geq 1$. Then $|D \tilde{g}|_{(j-1, j)} \geq 2$.
Case 2: Suppose that we are not in Case 1.
Case 2a: Suppose $j \neq 1$. Then since we are not in Case 1, there are no fixed points of $\tilde{g}$ in $[j, j-1]$. The only way to obtain fixed points is to take some iterate $\tilde{g}^{m}$ which passes through some interval $[i-1, i]$ which has $[j-1, j]$ in its image. The interval $(i-1, i)$ must be in Case 1, so we have $|D \tilde{g}|_{[i-1,1]} \geq 2$. Therefore, $\left|D \tilde{g}^{m}\right|_{(j-1, j)} \geq 2$.

Case 2b: Suppose $j=1$. If $d_{11}=1$ and $d_{1 i}=0$ for all $1<i \leq n$ then we proceed similarly to Case 2 a since we do not have any fixed points in $[0,1]$.

If $d_{11}=-1$ and $d_{i 1} \leq 1$ for some $1 \leq i \leq n$ then again we have $|D \tilde{g}|_{[0,1]} \geq 2$.
Therefore, in all cases for $x \in \operatorname{Fix}\left(\tilde{g}^{m}\right),\left|D \tilde{g}^{m}(x)\right|>1$.
Letting $g:=\pi \tilde{g} \pi^{-1}$, we are finished.
Note that we can often find some homotopic map $f$ which is also in $\mathcal{M}_{b}^{n}$.
Remark 15. Suppose that $f \in \mathcal{M}_{\# b}^{n}, f^{k}(b) \neq b$ for all $k \geq 1, d_{11}=-1$ and, contrary to Lemma 14, $d_{j 1}=0$ for all $1<j \leq n$. Then $f$ has two fixed points $x_{1}, x_{2}$ in $S_{1}$. It is easy see that a piecewise linear version on $f \mid S_{1}$ would have $\left|D f\left(x_{1}\right)\right|,\left|D f\left(x_{2}\right)\right|=1$, so this map could not be in $\mathcal{M}^{n}$. It is possible in some cases to perturb so that $\left|D f\left(x_{1}\right)\right|,\left|D f\left(x_{2}\right)\right|>1$, but this will always create some points $y \in S_{1}$ with $|D f(y)|<1$ which could mean that there are points $x \in G_{n}$ with $f^{m}(x)=x$ and $\left|D f^{m}(x)\right|<1$, i.e. $f \notin \mathcal{M}^{n}$.
Proof of Proposition 3. We first suppose that $f \in \mathcal{M}_{b}^{n}$. We consider the lift $\tilde{f}$. If $f$ is orientation preserving then

$$
\begin{equation*}
\# \operatorname{Fix}(f)=-1+\sum_{j=1}^{n} \chi_{j}\left(A_{j}\right) \tag{1}
\end{equation*}
$$

(We will explain this in our case, but it can also be seen for $G_{2}$ by looking at the proof of Proposition 2 of [10]). The reason for this is that for $1<j \leq n$, the image of $\tilde{f}([j-1, j])$ will start at $\tilde{f}(b)$ and, if there is some $i$ such that $a_{j_{k}}=a_{j}$, then this image must start below the diagonal $\{(x, x): 0 \leq x \leq n\}$ and cross it in order to cover $[j-1, j]$. This gives a fixed point every time this crossing happens.

When we are dealing with $\tilde{f}$ on $[0,1]$ we note that our map must miss the diagonal following the first appearance of $a_{1}$ in $A_{1}$. But for every subsequent appearance of $a_{1}$ there is a corresponding fixed point (the -1 term in (1) accounts for this).

If $f$ is orientation reversing then

$$
\begin{equation*}
\# \operatorname{Fix}(f)=1-\sum_{j=1}^{n} \chi_{j}\left(A_{j}\right) . \tag{2}
\end{equation*}
$$

This is essentially the same as the orientation preserving case except that any image $\tilde{f}([0,1])$ must cross the diagonal as many times as $a_{1}$ occurs in $A_{1}$, plus 1 . See for example Figure 2.

Clearly, given any $m \geq 1$, we can replace $A_{j}$ with $f_{\#}^{m}\left(a_{j}\right)$ in (1) or (2) to find \#Fix $\left(f^{m}\right)$ as required.

For $f \in \mathcal{M}_{b, k}^{n}$ where $k<\infty$ and $m \in \mathbb{N} \backslash k \mathbb{N}$ then the proof is the same as above. Now suppose that $m \in k \mathbb{N}$. For any $1 \leq j \leq n$, if $f_{\#}^{m}\left(a_{j}\right)$ has first or last element equal to $a_{j}$ then the graph of $\tilde{f}$ on $(j-1, j)$ has no corresponding crossing of the diagonal. However, if $a_{j}$ appears anywhere else in $f_{\#}^{m}\left(a_{j}\right)$ there is a corresponding crossing of the diagonal. Hence there are $\left|\gamma_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|$ fixed points of $f^{m}$ in $S_{j} \backslash\{b\}$. By assumption, there is also a fixed point of $f^{m}$ at $b$, so

$$
\# \operatorname{Fix}\left(f^{m}\right)=1+\sum_{j=1}^{n}\left|\gamma_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|
$$

as required.
By Proposition 3 the set of fixed points of $f \in \mathcal{M}^{n}$ are completely determined by the action on the fundamental group.

### 3.2. Finding periodic points from the fundamental group action.

Proof of Proposition 5. In all cases, $\left|d_{j j}\right| \geq 2$ for some $1 \leq j \leq n$, and so $f$ has a fixed point in $S_{j}$. We will show that in cases (a), (b), (c) and (d), when $m \geq 1$, \#Fix $\left(f^{m+1} \mid S_{j}\right)$, the number of fixed points of $f^{m+1}$ in $S_{j}$, is greater than $\# \operatorname{Fix}\left(f^{m} \mid S_{j}\right)$, the number of fixed points of $f^{m}$. Therefore, there must be some new fixed point of $f^{m+1}$, which has not been counted before as a fixed point for any $f^{p}$ where $p \leq m$. Hence we must have a periodic point of period $m+1$ in $S_{j}$. Since this will be true for any $m \geq 1$, we have $\operatorname{Per}(f)=\mathbb{N}$. In case (e) this argument will follow for any $m \geq 2$ and so $\operatorname{Per}(f) \supset \mathbb{N} \backslash\{2\}$.

Case 1: First suppose that $f \in \mathcal{M}_{b, k}^{n}$ where $k=1$, i.e. $f(b)=b$ and we are in case (d). Then we can see from the proof of Proposition 3 that the number of fixed points of $f^{p}$ in $S_{j}$ is $1+\left|\gamma_{j}\left(f_{\#}^{p}\left(a_{j}\right)\right)\right|$ (note that the 1 counts the fixed point at $b$ ). Therefore if we can show that

$$
\begin{equation*}
1+\left|\gamma_{j}\left(f_{\#}^{m+1}\left(a_{j}\right)\right)\right|>1+\left|\gamma_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right| \tag{3}
\end{equation*}
$$

then $\# \operatorname{Fix}\left(\left.f^{m+1}\right|_{S_{j}}\right)>\# \operatorname{Fix}\left(\left.f^{m}\right|_{S_{j}}\right)$ and there must be a periodic point of period $m+1$ in $S_{j}$.

Every element $a_{j}$ in the word $f_{\#}^{p}\left(a_{j}\right)$ gives rise to two occurrences of $a_{j}$ in $f_{\#}^{p+1}\left(a_{j}\right)$. Therefore, $\left|\gamma_{j}\left(f_{\#}^{p+1}\left(a_{j}\right)\right)\right| \geq 2\left|\gamma_{j}\left(f_{\#}^{p}\left(a_{j}\right)\right)\right|$, so (3) is satisfied whenever $\left|\gamma_{j}\left(f_{\#}^{p}\left(a_{j}\right)\right)\right|>1$. Since $\left|d_{j j}\right| \geq 2$ this is true for any $p>1$, so $\operatorname{Per}(f) \supset \mathbb{N} \backslash\{2\}$. For $p=1$ we have three cases: (i) if $\gamma_{j}\left(f_{\#}\left(a_{j}\right)\right)=0$ then $\left|\gamma_{j}\left(f_{\#}^{2}\left(a_{j}\right)\right)\right| \geq 2$, so $\# \operatorname{Fix}\left(f^{2} \mid S_{j}\right)>\# \operatorname{Fix}\left(\left.f\right|_{S_{j}}\right)$; (ii) if $\left|\gamma_{j}\left(f_{\#}\left(a_{j}\right)\right)\right|=1$ then $\left|\gamma_{j}\left(f_{\#}^{2}\left(a_{j}\right)\right)\right| \geq 3$, so $\# \operatorname{Fix}\left(f^{2} \mid S_{j}\right)>\# \operatorname{Fix}\left(\left.f\right|_{S_{j}}\right)$; (iii) if $\left|\gamma_{j}\left(f_{\#}\left(a_{j}\right)\right)\right|=2$ then $\left|\gamma_{j}\left(f_{\#}^{2}\left(a_{j}\right)\right)\right| \geq 4$, so $\# \operatorname{Fix}\left(f^{2} \mid S_{j}\right)>\# \operatorname{Fix}\left(\left.f\right|_{S_{j}}\right)$. Therefore, in all these cases, $\operatorname{Per}(f)=\mathbb{N}$.

From now on we will assume that $k>1$.
Case 2: We consider $m<k-1$. The proof here also follows when $f \in \mathcal{M}_{b}^{n}$. We can see from the proof of Proposition 3 that for $p \leq m$,

$$
-1+\left|\chi_{j}\left(f_{\#}^{p}\left(a_{j}\right)\right)\right| \leq \# \operatorname{Fix}\left(\left.f^{p}\right|_{S_{j}}\right) \leq 1+\left|\chi_{j}\left(f_{\#}^{p}\left(a_{j}\right)\right)\right|
$$

Since $\left|d_{j j}\right| \geq 2,\left|\chi_{j}\left(f_{\#}^{p+1}\left(a_{j}\right)\right)\right| \geq 2\left|\chi_{j}\left(f_{\#}^{p}\left(a_{j}\right)\right)\right|$ for any $p \geq 1$. Hence, we have

$$
\# \operatorname{Fix}\left(f^{m+1} \mid S_{j}\right) \geq-1+2\left|\chi_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right| \geq \# \operatorname{Fix}\left(f^{m} \mid S_{j}\right)+\left|\chi_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|-2 .
$$

Therefore, $\# \operatorname{Fix}\left(f^{m+1} \mid S_{j}\right)>\# \operatorname{Fix}\left(f^{m} \mid S_{j}\right)$ whenever $\left|\chi_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|>2$. This is always the case for $m \geq 2$.

For $m=1$ we have

$$
\begin{equation*}
\# \operatorname{Fix}\left(f^{2} \mid S_{j}\right) \geq-1+2\left|\chi_{j}\left(f_{\#}\left(a_{j}\right)\right)\right| . \tag{4}
\end{equation*}
$$

If we are in case (a) then we have $\# \operatorname{Fix}\left(\left.f\right|_{S_{j}}\right)=\left|\chi_{j}\left(f_{\#}\left(a_{j}\right)\right)\right|$, so by (4), $\# \operatorname{Fix}\left(\left.f^{2}\right|_{S_{j}}\right)>$ \#Fix $\left(\left.f\right|_{S_{j}}\right)$ and we are finished. For cases (b), (c) and (e), we have $j=1$. If we are in case (b) then $\# \operatorname{Fix}\left(\left.f\right|_{S_{1}}\right)=\left|\chi_{1}\left(f_{\#}\left(a_{1}\right)\right)\right|-1$, so by (4), $\# \operatorname{Fix}\left(f^{2} \mid S_{1}\right)>\# \operatorname{Fix}\left(\left.f\right|_{S_{1}}\right)$ and we are finished. If we are in case (c) then $\# \operatorname{Fix}\left(\left.f\right|_{S_{1}}\right)=\left|\chi_{1}\left(f_{\#}\left(a_{1}\right)\right)\right|+1$ and $\left|\chi_{1}\left(f_{\#}\left(a_{1}\right)\right)\right|>2$, so by (4), $\# \operatorname{Fix}\left(\left.f^{2}\right|_{S_{1}}\right)>\# \operatorname{Fix}\left(\left.f\right|_{S_{1}}\right)$ and we are finished. Note that case (d) is covered in Case 1. In case (e) it is possible that $\# \operatorname{Fix}\left(f^{2} \mid S_{S_{1}}\right)=\# \operatorname{Fix}\left(\left.f\right|_{S_{1}}\right)$, so we can't be sure if 2 is a period or not.

This proof also follows for the rest of the set $\mathbb{N} \backslash\{p: p=l k$ or $p=l k-1$ for $l \in \mathbb{N}\}$.
Case 3: We consider $m=l k-1$ for any $l \in \mathbb{N}$ where $k<\infty$. Considering the formulas for the number of fixed points of $f^{p}$ in $S_{j}$, it is sufficient to show that

$$
\begin{equation*}
1+\left|\gamma_{j}\left(f_{\#}^{m+1}\left(a_{j}\right)\right)\right|>\left|\chi_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|+1 \tag{5}
\end{equation*}
$$

We compute that for a word $b_{1} \ldots b_{n}$ allowed by $\mathcal{M}_{\#}^{n}, \gamma_{j}\left(b_{1} \ldots b_{n}\right) \geq\left|\chi_{j}\left(b_{1} \ldots b_{n}\right)\right|-2$. Therefore,

$$
\left|\gamma_{j}\left(f_{\#}^{m+1}\left(a_{j}\right)\right)\right| \geq 2\left|\chi_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|-2 .
$$

Hence, if $\left|\chi_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|>2$ then (5) is satisfied. Since $\left|\chi_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right| \geq 2^{m}$, (5) holds for case (c), or whenever $m>1$. For $m=1$, we compute that in case (a), $j=1$ and

$$
\begin{aligned}
\# \operatorname{Fix}\left(f^{2} \mid S_{1}\right) & =1+\left|\gamma_{1}\left(f_{\#}^{m+1}\left(a_{1}\right)\right)\right| \geq-1+\left|\chi_{1}\left(f_{\#}^{2}\left(a_{1}\right)\right)\right| \\
& \geq-1+2\left|\chi_{1}\left(f_{\#}\left(a_{1}\right)\right)\right|=\# \operatorname{Fix}\left(\left.f\right|_{S_{1}}\right)-1+\left|\chi_{1}\left(f_{\#}\left(a_{1}\right)\right)\right| .
\end{aligned}
$$

Since $\left|\chi_{1}\left(f_{\#}\left(a_{1}\right)\right)\right| \geq 2$ we are finished. In case (b), $j=1$

$$
\begin{aligned}
\# \operatorname{Fix}\left(f^{2} \mid S_{1}\right) & =1+\left|\gamma_{1}\left(f_{\#}^{m+1}\left(a_{1}\right)\right)\right| \geq-1+\left|\chi_{1}\left(f_{\#}^{2}\left(a_{1}\right)\right)\right| \\
& \geq-1+2\left|\chi_{1}\left(f_{\#}\left(a_{1}\right)\right)\right|=\# \operatorname{Fix}\left(\left.f\right|_{S_{1}}\right)+\left|\chi_{1}\left(f_{\#}\left(a_{1}\right)\right)\right| .
\end{aligned}
$$

Since $\left|\chi_{j}\left(f_{\#}\left(a_{j}\right)\right)\right| \geq 2$ we are finished.

Case 4: We consider $m=l k$ for any $l \in \mathbb{N}$ where $1<k<\infty$. Similarly to above, it is sufficient to show that

$$
\begin{equation*}
\left|\chi_{j}\left(f_{\#}^{m+1}\left(a_{j}\right)\right)\right|-1>\left|\gamma_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|+1 \tag{6}
\end{equation*}
$$

We have

$$
\left|\chi_{j}\left(f_{\#}^{m+1}\left(a_{j}\right)\right)\right| \geq 2\left|\chi_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right| \geq\left|\chi_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|+\left|\gamma_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|
$$

Therefore, (6) is satisfied whenever $\left|\chi_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|>2$. But this is always true when $k>1$.

Remark 16. As mentioned above, any map $f \in \mathcal{M}^{n}$ has a matrix action $\left(m_{i j}\right)$ on the first homology which has either all entries positive or all entries negative. Here the terms $m_{i j}$ take the place of $d_{i j}$. This was considered in [12] and with $b$ fixed in [11]. There the proof of the final part of Theorem 2 was proved applying Bolzano's Theorem to subgraphs.

Note that it is not the case that any map with such an action is homotopic to a map in $\mathcal{M}^{n}$. This is because the action on the first homology abelianises the action on the fundamental group. So, in particular, there exist homotopy classes with this action on the first homology, for which every map in the class has positive local degree at some point and negative local degree at some other point. (We say a map has positive (negative) local degree if the map is locally orientation preserving (reversing).)

Proof of Proposition 6. Suppose first that we are in case (a). We suppose that $\left|d_{j j}\right| \geq$ 1. Since we also have $\left|d_{i j}\right|,\left|d_{j i}\right| \geq 1$, we have $\left|\chi_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right| \geq 1$ for all $m \geq 0$ and $\left|\chi_{j}\left(f_{\#}^{m}\left(a_{i}\right)\right)\right| \geq 1$ for all $m \geq 1$. Note that in particular, $f$ has a fixed point in $S_{j}$.

Let $A_{j} \mapsto a_{1} a_{j} a_{i}$. (This is the simplest case for $j>1$, and, in terms of creating periodic points, the worst since $a_{j}$ only appears once in $A_{j}$.) Then we prove that every application of $f_{\#}$ to $f_{\#}^{m}\left(a_{j}\right)$ creates a new fixed point in $S_{j}$. We will use the fact that $f_{\#}$ is a homomorphism repeatedly.

We have $f_{\#}^{m+1}\left(a_{j}\right)=f_{\#}^{m}\left(a_{1}\right) f_{\#}^{m}\left(a_{i}\right) f_{\#}^{m}\left(a_{j}\right)$. The function $\chi_{j}$ counts the number of times $a_{j}$ appears in a given word. Thus, $\left|\chi_{j}\left(f_{\#}^{m+1}\left(a_{j}\right)\right)\right|=\left|\chi_{j}\left(f_{\#}^{m}\left(a_{1}\right) f_{\#}^{m}\left(a_{i}\right) f_{\#}^{m}\left(a_{j}\right)\right)\right| \geq$ $\left|\chi_{j}\left(f_{\#}^{m}\left(a_{j}\right)\right)\right|+1$ since $\left|\chi_{j}\left(f_{\#}^{m}\left(a_{i}\right)\right)\right| \geq 1$ for all $m \geq 0$. Therefore, the application of $f_{\#}$ generates a new fixed point in $S_{j}$. Whence $m \in \operatorname{Per}(f)$ for all $m \geq 1$. Note that if $A_{j}$ is a longer word, we obtain the same result (in that case, the number of fixed points created by each iteration could be even greater). Furthermore, if $f$ is orientation reversing we can apply the same proof. The proof of (d) follows similarly.

If we are in case (b) and not in case (a) or a case covered by Proposition 5 then we are in the orientation preserving case. The simplest form for $A_{1}$ is $a_{1} a_{i}$. We have $f_{\#}^{m+1}\left(a_{1}\right)=f_{\#}^{m}\left(a_{1}\right) f_{\#}^{m}\left(a_{i}\right)$. So again the fact that $\left|\chi_{1}\left(f_{\#}^{m}\left(a_{i}\right)\right)\right| \geq 1$ for all $m \geq 1$ means that we have found a new fixed point in $S_{1}$. Whence $m \in \operatorname{Per}(f)$ for all $m>1$.

If we are in case (c), then we find our new fixed points in $S_{i}$. We may suppose that $A_{1}=a_{1}^{-1} a_{i}^{-1}$ and $A_{i}=a_{1}^{-1}$. Then

$$
f_{\#}^{m+1}\left(a_{i}\right)=f_{\#}^{m}\left(a_{1}^{-1}\right)=f_{\#}^{m-1}\left(a_{1} a_{i}\right)=f_{\#}^{m-1}\left(a_{1}\right) f_{\#}^{m-1}\left(a_{i}\right)
$$

Since $\left|\chi_{i}\left(f_{\#}^{m}\left(a_{1}\right)\right)\right| \geq 1$ for $m \geq 1$, we find a new fixed point in $S_{i}$ after the application of $f^{2}$. Furthermore, each subsequent application of $f^{2}$ yields a new fixed point.

See Example 29 for an application of this. The following is an easy corollary of Proposition 6.

Proposition 17. Suppose that there exists some $m>1$ such that for $f \in \mathcal{M}_{\#}^{n}$, replacing $d_{i j}$ with $\chi_{i}\left(f_{\#}^{m}\left(a_{j}\right)\right)$ and $\mathcal{M}_{b, 1}^{n}$ by $\mathcal{M}_{b, m}^{n}$ in (d), we satisfy the conditions of Proposition 6. Then we have the same conclusions when we replace $\mathbb{N}$ with $m \mathbb{N}$ (in (b), the conclusion is replaced by $\operatorname{Per}(f) \supset m(\mathbb{N} \backslash\{1\}))$.

See Example 31 for an application of this.
Remark 18. Note that given a map $f \in \mathcal{M}_{b}^{n}$, this map has the minimal number of periodic points within the class of maps which are homotopic to $f$ and which have $b$ non-periodic (it is shown in [11] that the maps which minimise the number of fixed points within this homotopy class have $b$ fixed). For example, if $a_{1} \mapsto a_{1} a_{3} a_{1} a_{2} a_{2}$, then $f$ must cross the diagonal the number of times $a_{1}$ appears in the action minus 1 (minus one because $f(b) \in S_{1}$ implies that we go from $f(b)$ to $f(b)$ in $S_{1}$ without crossing the diagonal exactly once). But this is precisely what our maps do, and no more. We can argue similarly for $f \in \mathcal{M}_{b, k}^{n}$.

## 4. Periodic points and entropy

The main aim of this section is to prove Theorem 8. This involves showing that the eigenvalues of the matrices $f_{* 1}^{m}$ give us a lot of information about periodic points and about entropy. We first give entropy in terms of a limit involving $f_{* 1}^{m}$, proving Theorem 8(a) and then, for part (b), we give entropy in terms of the spectral radius of $f_{* 1}$.

We will give some basic definitions for entropy, see, for example [1] for more details. Let $X$ be a compact Hausdorff metric space. We say that the set $\mathcal{A}$ is an open cover for $X$ if $\cup_{A \in \mathcal{A}} A \supset X$ and all $A$ are open sets. A subcover of $X$ from $\mathcal{A}$ is a subset of $\mathcal{A}$ which is also a cover of $X$. When it is clear what $X$ is, we simply refer to covers and subcovers.

Let $\mathcal{A}$ be an open cover of $X$. For a continuous map $f: X \rightarrow X$, we define $f^{-i}(\mathcal{A}):=$ $\left\{f^{-i}(A): A \in \mathcal{A}\right\}, \bigvee_{i=1}^{m-1} \mathcal{A}_{i}:=\left\{A_{1} \cap \ldots \cap A_{m-1}: A_{i} \in \mathcal{A}_{i}\right.$, and $\left.A_{1} \cap \ldots \cap A_{m-1} \neq \emptyset\right\}$ and $\mathcal{A}^{m}:=\bigvee_{i=0}^{m-1} f^{-i}(\mathcal{A})$. Also let $\mathcal{N}(\mathcal{A})$ be the minimal cardinality of any subcover from $\mathcal{A}$.

Let

$$
h(f, \mathcal{A}):=\lim _{m \rightarrow \infty} \frac{1}{m} \log \mathcal{N}\left(A^{m}\right) .
$$

Then we define the topological entropy of $f$ to be

$$
h(f):=\sup h(f, \mathcal{A})
$$

where the supremum is taken over all open covers of $X$.
Here we will let $X$ be some bouquet $G_{n}$. We say that $\mathcal{A}$ is a cover of $G_{n}$ by arcs if $\cup_{A \in \mathcal{A}} A \supset G_{n}$, each $A \in \mathcal{A}$ is an arc of $G_{n}$ and all $A \in \mathcal{A}$ are pairwise disjoint. These arcs can be open or closed or half open and half closed or even degenerate. (This notion is similar to 'a cover by intervals' when the phase space is the interval, see Chapter 4.2 of [1].)

Let $\mathcal{A}$ be a cover of $G_{n}$. We call $\mathcal{A}$ an $f$-mono cover if for all $A_{i} \in \mathcal{A}$ there is some circle $S_{j}$ such that $f: A_{i} \rightarrow S_{j}$ is an injective homeomorphism. Note that if $\mathcal{A}$ is an $f$-mono cover then $\mathcal{A}^{m}$ is an $f^{m}$-mono cover.

The following results will allow us to prove Theorem 8(a). Propositions 19 and 21 are adapted versions of the theory of [15]. We follow the exposition of this theory in [1].

Proposition 19. For $f \in \mathcal{M}_{n}, h(f)=\sup (f, \mathcal{A})$ where the supremum is taken over finite covers by arcs.

For the proof of this see Proposition 4.2.2 of [1] which proves that this is so for interval maps and finite covers by intervals.
Lemma 20. Suppose that $f \in \mathcal{M}^{n}$. Suppose that $M$ is the matrix $f_{* 1}$. Then there is a natural $f$-mono cover by arcs with cardinality $\|A\|$.
Proof. We construct the cover as follows. Considering the lift $\tilde{f}:[0, n] \rightarrow[0, n]$, let $P=\tilde{f}^{-1}(b)=\left\{p_{1}, \ldots p_{m}\right\}$ where $p_{1}<\cdots<p_{m}$. For $1 \leq i<m$, let $P_{i}=\left[p_{i}, p_{i+1}\right)$. Also, let $P_{m}=\left[p_{m}, n\right] \cup\left[0, p_{1}\right)$. Let $A_{i}=\pi\left(P_{i}\right)$ (note that $\pi\left(P_{m}\right)$ is an arc) and let $\mathcal{A}=\left\{A_{1}, \ldots, A_{m}\right\}$.

Since $\#(\mathcal{A})=\#\left(\tilde{f}^{-1}(b)\right)$, we have $\#(\mathcal{A})=\|A\|$ as required.
The following is Proposition 4.2.3 of [1] with minor adaptations so that it applies to our case (we must adapt the situation for interval maps to the situation for maps on bouquets). We include a proof for completeness.

Proposition 21. For $f \in \mathcal{M}^{n}$, if $\mathcal{A}$ is a mono-cover of $G_{n}$ then $h(f)=h(f, \mathcal{A})$.
Proof. Let $\tilde{\mathcal{B}}$ be a finite cover of $G_{n}$ by arcs. Let $\mathcal{B}=\tilde{\mathcal{B}} \vee \mathcal{A}$. Let $\mathcal{C}$ be a cover chosen from $\mathcal{A}^{m}$. Take $A \in \mathcal{C}$. The map $\left.f^{k}\right|_{A}$ is a homeomorphism for $k=1, \ldots, m$. Therefore, for any $B \in \mathcal{B}$, the set $A \cap f^{-k}(B)$ is an arc (unless it is empty). Each arc has at most 2 endpoints (note that a degenerate arc has only one endpoint). Let $x$ be an endpoint of an element $B \in \mathcal{B}^{m}$. Then there exists $A \in \mathcal{C}$ such that $A \cap B \neq \emptyset$ and $x$ is a endpoint of $A \cap B$. Since $B=\bigcap_{k=0}^{m-1} f^{-k}\left(B_{k}\right)$ for some $B_{0}, \ldots, B_{m-1} \in \mathcal{B}$ and each of the sets $f^{-k}\left(B_{k}\right)$ is a union of a finite number of arcs, $x$ is an endpoint of some component of $f^{-k}\left(B_{k}\right)$ for some $k \in\{0, \ldots, m-1\}$. Hence $x$ is an endpoint of $A \cap f^{-k}(B)$ for this $k$. In each $A \in \mathcal{C}$ there are at most $2 m \#(\mathcal{B})$ such endpoints. The number of possible arcs with endpoints in a given set is not larger than 4 times the square of the cardinality of this set (we multiply by 4 because arcs with given endpoints may or may not contain them $)$. Therefore, $\#\left(\left.\mathcal{B}^{m}\right|_{A}\right) \leq 4\left(2 m \#\left(\mathcal{B}^{m}\right)\right)^{2}$. Hence, $\mathcal{N}\left(\mathcal{B}^{m}\right) \leq 4\left(2 m \#\left(\mathcal{B}^{m}\right)\right)^{2} \#(\mathcal{C})$. Since $\mathcal{C}$ was arbitrary, we obtain

$$
\mathcal{N}\left(\mathcal{B}^{m}\right) \leq 4\left(2 m \#\left(\mathcal{B}^{m}\right)\right)^{2} \mathcal{N}\left(\mathcal{A}^{m}\right)
$$

In the limit we get

$$
h(f, \tilde{\mathcal{B}}) \leq h(f, \mathcal{B}) \leq h(f, \mathcal{A})
$$

By Proposition 19, in calculating the entropy we need only consider finite covers by arcs, so we have $h(f) \leq h(f, \mathcal{A})$, and consequently $h(f)=h(f, \mathcal{A})$.

The following is proved in the appendix of [14].
Lemma 22. For a matrix of complex numbers $M$, the limit $\lim _{m \rightarrow \infty} \log \left\|M^{m}\right\|^{\frac{1}{m}}$ exists.
Proof of Theorem 8(a). Consider the $f$-mono cover $\mathcal{A}$ of $G_{n}$ constructed in Lemma 20. We let $M$ be the action of $f_{* 1}$ on the first homology. Then $\#(\mathcal{A})=\|M\|$. Furthermore, $\#\left(\mathcal{A}^{m}\right)=\left\|M^{m}\right\|$. Therefore, by Proposition 21,

$$
h(f)=h(f, \mathcal{A})=\lim _{m \rightarrow \infty} \frac{1}{m} \log \left\|M^{m}\right\|
$$

Since, by Lemma 22 this limit exists (we could also refer to Section 4.1 of [1] to show that any such limit of the cardinality of the pullback of covers exists), Theorem 8(a) is proved.

The proof of Theorem 8(b) is a simple corollary of Theorem 8(a) and the following result: Theorem A. 3 of [14]. The proof also follows from [2].

Theorem 23. The spectral radius of any real or complex $n \times n$ matrix $M$ is given by

$$
\sigma(M)=\lim _{k \rightarrow \infty}\left\|M^{k}\right\|^{\frac{1}{k}}=\lim \sup _{k \rightarrow \infty}\left\|\operatorname{Tr}\left(M^{k}\right)\right\|^{\frac{1}{k}}
$$

For applications of Theorem 8, see any of the examples in Section 6.

## 5. Computing periods from eigenvalues of $f_{* 1}$

As above, the spectral radius of $f_{* 1}$ can be computed as $\lim \sup _{k \rightarrow \infty}\left\|\operatorname{Tr}\left(M^{k}\right)\right\|^{\frac{1}{k}}$. However, if $\lim _{k \rightarrow \infty}\left\|\operatorname{Tr}\left(M^{k}\right)\right\|^{\frac{1}{k}}$ exists then we can say more. We first state a result of [6] (in fact, there the theorem also extends to maps with higher homologies than we consider here). We need the following definition. A $C^{1} \operatorname{map} f: M \rightarrow M$ of a compact $C^{1}$ differentiable manifold is called transversal if $f(M) \subset M$ and for all $m \in \mathbb{N}$, for all $x \in \operatorname{Fix}\left(f^{m}\right)$ we have $\operatorname{det}\left(I-d f^{m}(x)\right) \neq 0$, i.e. 1 is not an eigenvalue of $d f^{m}(x)$.

Theorem 24. Let $M$ be a compact manifold with $H_{i}(M, \mathbb{Q})=0$ for $i>1$. Suppose that $f: M \rightarrow \operatorname{Int}(M)$ is a $C^{1}$ transversal map. Further, assume that the limits

$$
\lim _{m \rightarrow \infty}\left|\operatorname{Tr}\left(f_{* 1}^{m}\right)\right|^{\frac{1}{m}}
$$

and

$$
\lim _{m \rightarrow \infty}\left|\sum_{d \mid m} \mu(d) \operatorname{Tr}\left(f_{* 1}\right)\right|^{\frac{1}{m}}
$$

exist. If there is an eigenvalue different from a root of unity or zero then there exists $m_{0} \geq 1$ such that
(a) for all $m \geq m_{0}$ odd we have that $m \in \operatorname{Per}(f)$;
(b) for all $m \geq m_{0}$ even we have that $\left\{\frac{m}{2}, m\right\} \cap \operatorname{Per}(f) \neq \emptyset$.

Remark 25. Suppose that $\left|\lambda_{1}\right|>1$ and $\left|\lambda_{1}\right|>\left|\lambda_{2}\right|$. Then we claim that the limit $\lim _{m \rightarrow \infty}\left|\operatorname{Tr}\left(f_{* 1}^{m}\right)\right|^{\frac{1}{m}}$ exists and is equal to $\left|\lambda_{1}\right|$ since

$$
\left|\lambda_{1}\right|^{k}-(d-1)\left|\lambda_{2}\right|^{k}<\left|\operatorname{Tr}\left(f_{* 1}^{k}\right)\right|<\left|\lambda_{1}\right|^{k}+(d-1)\left|\lambda_{2}\right|^{k}
$$

Taking limits we prove the claim.
In a similar way we are able to show that

$$
\lim _{m \rightarrow \infty}\left|\sum_{d \mid m} \mu(d) \operatorname{Tr}\left(f_{* 1}\right)\right|^{\frac{1}{m}}
$$

exists. This is because it can be shown that there exists some $C>0$ such that

$$
\left|\lambda_{1}\right|^{k}-\frac{1}{C} m\left|\lambda_{1}\right|^{\frac{k}{2}}<\left|\sum_{d \mid m} \mu(d) \operatorname{Tr}\left(f_{* 1}\right)\right|<\left|\lambda_{1}\right|^{k}+C m\left|\lambda_{1}\right|^{\frac{k}{2}}
$$

(For a calculation of this type, see the proof of Proposition 9.) Letting $k \rightarrow \infty$ we again obtain $\left|\lambda_{1}\right|$ as the limit. Therefore we have the conclusions of Theorem 24 for our map.

In fact, in our class, we can improve this result to obtain Proposition 9. For our proof, we need to show that if a particular growth condition on the number of fixed points is satisfied then we can be sure of the existence of some periodic points. To give an idea of this approach we state the following easily proved claim.

Claim 26. Suppose that $f: M \rightarrow M$ is some map on some space $M$. If we have

$$
\# \operatorname{Fix}_{m}(f)>\sum_{r \mid m, r \neq m} \# \operatorname{Per}_{r}(f)
$$

then

$$
\# \operatorname{Per}\left(f^{m}\right)=\# \operatorname{Fix}_{m}(f)-\sum_{r \mid m, r \neq m} \# \operatorname{Per}_{r}(f)>0
$$

Now we give the main tool for the proof of Proposition 9.
Proposition 27. Suppose that $f: M \rightarrow M$ is some map on some space $M$. If for some $m \geq 1$,

$$
\# \operatorname{Fix}\left(f^{m}\right)>\sum_{\frac{m}{k} \text { prime }, k \neq m} \# \operatorname{Fix}\left(f^{k}\right)
$$

then $\operatorname{Per}_{m}(f) \neq \emptyset$.
Proof. We have

$$
\# \operatorname{Fix}\left(f^{m}\right)=\sum_{r \mid m} \# \operatorname{Per}_{r}(f)
$$

Now supposing that $\operatorname{Per}_{m}(f)=\emptyset$,

$$
\# \operatorname{Fix}\left(f^{m}\right)=\sum_{r \mid m, r \neq m} \# \operatorname{Per}_{r}(f)
$$

So if we prove that

$$
\sum_{\frac{m}{k} \text { prime }, k \neq m} \# \operatorname{Fix}\left(f^{k}\right) \geq \sum_{r \mid m, r \neq m} \# \operatorname{Per}_{r}(f)
$$

then the proposition will follow.
Note that we can write $m$ as a product of prime factors $m=p_{1} \ldots p_{n}$. Thus, if $\frac{m}{k}$ is prime and $k \neq m$ then $k=\frac{p_{1} \ldots p_{n}}{p_{i}}$ for some $1 \leq i \leq n$. So

$$
\# \operatorname{Fix}\left(f^{k}\right)=\# \operatorname{Fix}\left(f^{\frac{p_{1} \ldots p_{n}}{p_{i}}}\right)=\sum \# \operatorname{Per}_{r}(f)
$$

where the sum runs over all combinations $r=p_{j_{1}} \cdots p_{j_{n_{r}}}$ where $j_{1}<\cdots<j_{q}$ and all $j_{k} \in\{1, \ldots, n\} \backslash\{i\}$.

We can express any $r \mid m$ which has $r \neq m$ as prime factors: $r=p_{j_{1}} \ldots p_{j_{n_{r}}}$ where $1 \leq n_{r}<n$. Therefore, the term $\# \operatorname{Per}_{r}(f)$ is counted $\binom{n-1}{n_{r}}(\geq 1)$ times by the sum on the left, but only once by the sum on the right. So the proposition is proved.

Proof of Proposition 9. First suppose that $f \in \mathcal{M}_{b}^{n}$. We assume that for some $m>1$, $\operatorname{Per}_{m}(f)=\emptyset$. Otherwise we are finished. From Proposition 27, to prove that we contradict this assumption on periodic points, it is sufficient to show that

$$
\begin{equation*}
\# \operatorname{Fix}\left(f^{m}\right)>\sum_{\frac{m}{k} \text { prime }, k \neq m} \# \operatorname{Fix}\left(f^{k}\right) . \tag{7}
\end{equation*}
$$

By Theorem 2, for any $m \geq 1, \# \operatorname{Fix}\left(f^{m}\right)=\left|L\left(f^{m}\right)\right|=\left|1-\left(\lambda_{1}^{m}+\cdots+\lambda_{d}^{m}\right)\right|$. Clearly, for $m \geq 2$,

$$
m\left(\left|\lambda_{1}\right|^{\frac{m}{2}}+\cdots+\left|\lambda_{d}\right|^{\frac{m}{2}}-1\right)>\sum_{\frac{m}{k} \text { prime, } k \neq m}\left|1-\left(\lambda_{1}^{k}+\cdots+\lambda_{d}^{k}\right)\right|
$$

But if $m$ is large enough then,

$$
\begin{equation*}
\left|\lambda_{1}^{m}+\cdots+\lambda_{d}^{m}\right|-1>m\left(\left|\lambda_{1}\right|^{\frac{m}{2}}+\cdots+\left|\lambda_{d}\right|^{\frac{m}{2}}+1\right) \tag{8}
\end{equation*}
$$

which is sufficient to give (7). For example we have the inequalities $\left|1-\left(\lambda_{1}^{m}+\cdots+\lambda_{d}^{m}\right)\right|>$ $\left|\lambda_{1}\right|^{m}-(d-1)\left|\lambda_{2}\right|^{m}-1$ and $m\left(1+\left|\lambda_{1}\right|^{\frac{m}{2}}+\cdots+\left|\lambda_{d}\right|^{\frac{m}{2}}\right)<m\left(d\left|\lambda_{1}\right|^{\frac{m}{2}}+1\right)$, so whenever,

$$
\left|\lambda_{1}\right|^{m}-(d-1)\left|\lambda_{2}\right|^{m}-1>m\left(d\left|\lambda_{1}\right|^{\frac{m}{2}}+1\right)
$$

then (8) is satisfied and the proposition is proved for $f \in \mathcal{M}_{b}^{n}$. When $f \in \mathcal{M}_{b, k}^{n}$ for $k<\infty$ and $m \in k \mathbb{N}$ then Remark 4 gives $L\left(f^{m}\right) \leq \# \operatorname{Fix}\left(f^{m}\right) \leq 2 n-1+L\left(f^{m}\right)$. Since when $m$ is large, the $2 n-1$ term becomes insignificant in terms of the size of $\left|\lambda_{1}\right|^{m}$, we see that we can apply the same proof as above to this case too.

See Examples 29 and 32 for applications of this. Note that there are many examples where the condition $\left|\lambda_{1}\right|>1$ and $\left|\lambda_{1}\right|>\left|\lambda_{2}\right|$ is not satisfied, but we still have $\operatorname{Per}(f)=$ $\mathbb{N}$. For example, consider a map in $\mathcal{M}_{b}^{2}$ which has the action on the first homology of a matrix with 2's on the diagonal and zeros elsewhere.

Note that in Example 31 we have a situation where there are eigenvalues of $f_{* 1}$ which are strictly greater than 1 , but $\operatorname{Per}(f)=3 \mathbb{N}$. So there are limits to how far we can extend this result.

Remark 28. We would like to estimate $m_{0}$ for $f \in \mathcal{M}_{b}^{n}$. From the above proof, we require that $m_{0}$ is the infimum of all $m$ such that

$$
\left|\lambda_{1}\right|^{m}>d\left[m\left(\left|\lambda_{1}\right|^{\frac{m}{2}}+1\right)+\left|\lambda_{2}\right|^{m}\right]+1
$$

## 6. ExAMPLES

We may apply our results to the following examples.
Example 29. Suppose that $f \in \mathcal{M}_{b}^{3}$ and $f_{* 1}$ has matrix

$$
\left(\begin{array}{lll}
1 & 1 & 1 \\
0 & 0 & 0 \\
1 & 0 & 1
\end{array}\right)
$$

This is a matrix satisfying the conditions of Proposition 6 and so has $\operatorname{Per}(f)=\mathbb{N}$. Indeed, for $m>1$,

$$
f_{* 1}^{m}=\left(\begin{array}{rrr}
2^{m-1} & 2^{m-2} & 2^{m-1} \\
0 & 0 & 0 \\
2^{m-1} & 2^{m-2} & 2^{m-1}
\end{array}\right)
$$

so we have exponential growth of the trace. Furthermore, the eigenvalues of $f_{* 1}$ are 0,0 and 2 . So by Theorem 8 , the entropy is $\log 2$.

The following example has entropy zero.

Example 30. Suppose that $f \in \mathcal{M}_{b}^{4}$ where $f_{* 1}$ has action

$$
\left(\begin{array}{llll}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right)
$$

Then we look at the first six iterates of this matrix:

$$
\begin{aligned}
& \left(\begin{array}{llll}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right),\left(\begin{array}{llll}
1 & 2 & 2 & 2 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{array}\right),\left(\begin{array}{llll}
1 & 3 & 3 & 3 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right), \\
& \left(\begin{array}{llll}
1 & 4 & 4 & 4 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{array}\right),\left(\begin{array}{llll}
1 & 5 & 5 & 5 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0
\end{array}\right),\left(\begin{array}{llll}
1 & 6 & 6 & 6 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right) .
\end{aligned}
$$

In fact, we see that if $m=3 k$ for $k \geq 1$, then

$$
f_{* 1}^{m}=\left(\begin{array}{cccc}
1 & m & m & m \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{array}\right)
$$

and if $3 \nmid m$ then the only non-zero entry on the diagonal is the top left corner. Thus $L\left(f^{m}\right)=0$. If $3 \mid m$ then $L\left(f^{m}\right)=-3$. $L(f), L\left(f^{2}\right)=0$. If $m=3$ then $l\left(f^{3}\right)=2$. But when $m>3$,

$$
l\left(f^{m}\right)=\sum_{\substack{r|m \\ 3| \frac{m}{r}}} \mu(r) L\left(f^{\frac{m}{r}}\right)
$$

So clearly, if $3 \nmid m$ then $l(f)=0$, and if $3 \mid m$ then

$$
L\left(f^{m}\right)=-3 \sum_{r \left\lvert\, \frac{m}{3}\right.} \mu(r)=0 .
$$

(For more details of this last calculation, see for example Section 3 of [7].) Therefore, $\operatorname{Per}(f)=\{3\}$.

Also, we calculate that the eigenvalues of this matrix are $1,-1, e^{\frac{i \pi}{3}}, e^{-\frac{i \pi}{3}}$, and find $l\left(f^{m}\right)$ this way. By Theorem 8, the entropy of this system is zero (which we would expect since there is no growth of periodic points).

We see in the next two examples that a small change to the matrix in Example 30 can alter the entropy and the growth of periodic points.

Example 31. Now instead consider the matrix in Example 30, but with any one of the entries $m_{i j}$ for $i, j>1$ which equalled 1 , replaced by 2 . Then the eigenvalues of this matrix are $1,-\left|2^{\frac{1}{3}}\right|,\left|2^{\frac{1}{3}}\right| e^{\frac{i \pi}{3}},\left|2^{\frac{1}{3}}\right| e^{-\frac{i \pi}{3}}$. By Theorem 8 , the entropy is $\frac{\log 2}{3}$ and $\operatorname{Per}(f)=3 \mathbb{N}$. We can see this, for example, by applying Proposition 17 to $f^{3}$ (since we can compute that the entry in the bottom right-hand corner of the matrix $f_{* 1}^{3}$ is 2 ). We could also show this by direct calculation.

Note that we cannot apply Proposition 9 here since $\left|\lambda_{1}\right|=\left|\lambda_{2}\right|$.

Example 32. Consider $f \in \mathcal{M}_{b}^{4}$ with action

$$
f_{* 1}=\left(\begin{array}{cccc}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1
\end{array}\right)
$$

We calculate that

$$
f_{* 1}^{2}=\left(\begin{array}{cccc}
1 & 2 & 2 & 3 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 1 & 1
\end{array}\right), f_{* 1}^{3}=\left(\begin{array}{cccc}
1 & 3 & 4 & 6 \\
0 & 1 & 1 & 1 \\
0 & 0 & 1 & 1 \\
0 & 1 & 1 & 2
\end{array}\right)
$$

It is easy to see from these matrices that $\operatorname{Per}(f)=\mathbb{N} \backslash\{2\}$. Note that we have $m_{0}=3$ in the statement of Proposition 9. However, note that since the eigenvalues for $f_{* 1}$ are $\lambda_{1}=1.47, \lambda_{2}=1, \lambda_{3}=0.23+i 0.79, \lambda_{4}=0.23-i 0.79$, the calculation in Remark 28 gives $m_{0}=10$; which is far from optimal.

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